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COMPLETE NON-SELFADJOINTNESS OF ALMOST SELFADJOINT OPERATORS

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Suppose that α is a real-valued measurable function defined on the unit interval $[0, 1]$ and that c is a function in the Lebesgue space $L^2(0, 1)$. Let A be the (not necessarily bounded) operator on $L^2(0, 1)$ associated with the pair (α, c) by

$$(Af)(x) = \alpha(x)f(x) + i\, c(x)\int_0^x \overline{c(t)}\, f(t)dt.$$

A has the domain

$$\mathscr{D}(A) = \{f \in L^2(0, 1): \int_0^1 |\alpha(x)f(x)|^2\, dx < \infty\}$$

which is dense in $L^2(0, 1)$. One easily verifies that the imaginary part $(2i)^{-1}(A - A^*)$ extends to the bounded operator $f \to 1/2\langle f, c\rangle c$. Thus A is almost selfadjoint in the sense that it differs from its real part by an operator of rank one.

The operators A are more general than they appear. Livsic showed that every bounded operator B with real spectrum, no selfadjoint part, and with nonnegative imaginary part of rank one is unitarily equivalent to the completely non-selfadjoint part of such an operator A acting on $L^2(0, a)$ for some positive a. This raises the question of when (in terms of α and c) A is completely non-selfadjoint. The main result of this paper answers this question when the pair (α, c) is subject to a mild restriction that is always satisfied when A is bounded.

One consequence (Corollary 3.18) is a negative result concerning the behavior of singular spectral multiplicity under compact perturbations.

We need to establish some conventions and terminology. All Hilbert spaces throughout will be separable. Let B be a densely defined operator on a Hilbert space H with domain $\mathscr{D}(B)$. We will say that a subspace N of H *reduces* B if $\mathscr{D}(B) \cap N$ and $\mathscr{D}(B) \cap N^\perp$ are dense in N and $N^\perp$, respectively, and $B(\mathscr{D}(B) \cap N) \subset N$ and $B(\mathscr{D}(B) \cap N^\perp) \subset N^\perp$. B is said to be *completely non-selfadjoint* if the only reducing subspace N for B with the property that the restriction $B|N$ is selfadjoint is the zero subspace.

B is *dissipative* if $\operatorname{Im}\langle Bf, f\rangle \geqq 0$ for all f in $\mathscr{D}(B)$. If in addition $(B + i/2)\mathscr{D}(B) = H$, then B is called *maximal dissipative*. In this case the Cayley transform $C = (B - i/2)(B + i/2)^{-1}$ is a contraction defined on all of H. (We have replaced i by $i/2$ in the Cayley