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ON A CONJECTURE OF L. B. PAGE

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If $\mathscr{K}$ is an Hilbert space and $\mathscr{M}$ a subspace which is invariant under a unilateral shift S on $\mathscr{K}$ one can ask when a bounded operator T on $\mathscr{M}$ which commutes with S can be extended to a bounded operator on all of $\mathscr{K}$ which also commutes with S. Here this problem is considered in the special case that $\mathscr{K}$ is a Hardy space H^2 of functions analytic in the unit-disk with values in a finite dimensional Hilbert space. For this situation an easily derived necessary condition is shown to be sufficient. Further those $\mathscr{M}$ for which the extension to $\mathscr{K}$ is unique are characterized.

The problem posed above has recently been considered by Lavon B. Page [4], who conjectured that the condition (2) below was sufficient as well as necessary and who proved the validity of this conjecture in several cases.

Throughout this note we shall use facts, mainly elementary or standard, about unilateral shifts with few direct references given. These facts can all be found in [1.2.3]. A subspace will alwas be a closed linear manifold and all operators will be linear and continuous.

Let $\mathscr{K}_n$ denote the Hardy space of H^2 functions in the unit disk $\Delta = \{z \mid |z| < 1\}$ with values in the complex Hilbert space C_n, $(n < +\infty)$. When convenient $u \in \mathscr{K}_n$ will be considered as an $n \times 1$ column vector of elements in $\mathscr{K}_1 = H^2$, and we shall freely identify $\mathscr{K}_p \oplus \mathscr{K}_q$ with $\mathscr{K}_{p+q}$. Let S denote the unilateral shift on $\mathscr{K}_n$ generated by multiplication by z, i.e., $(Su)(z) = zu(z)$ for all $z \in \Delta$ and $u \in \mathscr{K}_n$. Then $S^*S = I$ but $SS^* \neq I$. For each positive integer m put $P_m = I - S^m S^{*m}$. Then P_m is an orthogonal projection and for any operator T commuting with S, $P_m T = P_m T P_m$. Thus for all $u \in \mathscr{K}_n$ we have,

$$\|P_m Tu\| = \|P_m T P_m u\| \leqq \|T\| \, \|P_m u\| . \tag{1}$$

If $\mathscr{M}$ is a subspace of $\mathscr{K}_n$ which is invariant under S and T is an operator on $\mathscr{M}$ which commutes with S and has a continuous extension to all of $\mathscr{K}_n$ which also commutes with S, then clearly this extension must satisfy (1) on $\mathscr{K}_n$. Hence T itself must satisfy (1) on $\mathscr{M}$. This lead Page to the following conjecture [4]:

If $\mathscr{M}$ is a subspace of $\mathscr{K}_n$, invariant under S, and T an operator on $\mathscr{M}$ which commutes with S and satisfies (2) $\|P_m Tu\| \leqq \alpha \|P_m u\|$ for all $u \in \mathscr{M}$ and m, then T has an extension to all of $\mathscr{K}_n$ which commutes with S and has norm less than or equal to α.

We have taken the liberty here of reducing Page's conjecture