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A SEPARABLY CLOSED RING WITH NONZERO TORSION PIC

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We give an example of a ring and a rank one projective module over that ring such that the square of the module is free but the module does not become free over any separable extension of the ring.

Every ideal class in the ring of integers in a number field can be split by an unramified extension. Over a commutative ring which is an algebra over the rationals every torsion element of Pic of the ring is split by a separable extension [3]. These examples suggest the question: is the torsion part of Pic of a separably closed ring trivial? We will exhibit a ring which shows the answer is negative. The ring arises as a slight modification of an example of Swan [5].

For any commutative ring k, k^X denotes the group of units of k, Cl(k) denotes the divisor class group if k is a domain, and $Qu(k)$ denotes the group of quadratic extensions of k. We use Z for the integers and Q for the rationals.

DEFINITION. For any commutative ring k, let $k(S^1) = k[X_0, X_1]/(X_0^2 + X_1^2 - 1)$. Let t_i be the image of X_i in $k(S^1)$. $k(S^1)$ is graded mod 2; let $k(P^1)$ be the even graded piece and $L(k)$ the odd.

LEMMA 1. *$L(k)$ is a projective $k(P^1)$-module of rank 1 whose tensor square is free.*

Proof. It suffices to check the first assertion for $k = Z$. By the argument in [5, p. 271] $L(Z)$ is projective of rank 1. The multiplication in $k(S^1)$ defines a homomorphism

$$L(k) \otimes_{k(P^1)} L(k) \rightarrow k(P^1)$$

whose image contains $t_0^2 + t_1^2 = 1$ and is thus an isomorphism.

We will show that $L(Z)$ cannot be split by a separable extension of $Z(P^1)$. We begin by collecting some facts about the rings involved.

LEMMA 2. (a) *$Q(i)(P^1) = Q(i)[v, v^{-1}]$ where $v = (t_0 + it_1)^2$*
(b) $Q(P^1)^X = Q^X$
(c) *$Z/2Z(P^1)$ is a polynomial ring (in one variable) over $Z/2Z$*
(d) *$L(Z/2Z)$ is freely generated by $t_0 + t_1$*

Proof. Let $K = Q(i)$. Then $K(S^1) = K[u, u^{-1}]$ where $u = t_0 + it_1$.