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ON ABSOLUTE DE LA VALLÉE POUSSIN SUMMABILITY

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Gronwall proved that $(C, r) \subseteqq (V - P)$ for $r \geqq 0$, where (C, r) and $(V - P)$ denote Cesáro and de la Vallée Poussin summability. It is proved in this paper that $|C, r| \subseteqq |V - P|$ for $r \geqq 0$.

1. Introduction. Let

$$V_n = \sum_{k=1}^{n} \frac{(n!)^2}{(n-k)!(n+k)!} a_k \quad (n \geqq 0) .$$

If $\lim_{n\to\infty} V_n = s$, we say that the series is summable $(V - P)$ to s. If

$$\sum_{n=1}^{\infty} \left| V_n - V_{n-1} \right| < \infty .$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|V - P|$.

Hyslop [2] proved that the $(V - P)$ method is equivalent to the $(A, 2)$ method defined by

$$\lim_{x\to 0+} \sum_{n=0}^{\infty} a_n e^{-n^2 x} = s$$

for all series $\sum_{n=0}^{\infty} a_n$ which satisfy the condition $a_n = 0(n^c)$, where c is any constant, and that the inclusion $(A, 2) \subseteqq (V - P)$ is false without restriction.

Kuttner [3] has shown that $(V - P) \subseteqq (A, 2)$ without restriction.

Gronwall [1] proved that $(C, r) \subseteqq (V - P)$ for $r \geqq 0$, where (C, r) denotes the Césaro summability of order r.

In this paper, we shall prove

THEOREM A. *$|C, r| \subseteqq |V - P|$ for $r \geqq 0$.*

2. Proof of Theorem A. Since it is well-known that $|C, r|$ implies $|C, r'|$ for $-1 < r \leqq r'$, it is enough to consider the case r an integer. Now, writing

$$V_n = v_0 + v_1 + \cdots + v_n ,$$

we find that

$$(1) \qquad \begin{cases} v_0 = a_0 , \\ v_n = \sum_{k=1}^{n} \frac{((n-1)!)^2}{(n-k)!(n+k)!} k^2 a_k \quad (n \geqq 1) . \end{cases}$$