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GENERALIZED QUASICENTER AND HYPERQUASICENTER OF A FINITE GROUP

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The notion of quasicentral element is generalized to p-quasicentral element and the p-quasicenter and the p-hyperquasicenter are defined. It is shown that the p-quasicenter is p-supersolvable and the p-hyperquasicenter is p-solvable.

The quasicenter $Q(G)$ of a group G is the subgroup of G generated by all quasicentral elements of G, where an element x of G is called a quasicentral element (QC-element) when the cyclic subgroup $\langle x\rangle$ generated by x satisfies $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for all elements y of G. The hyperquasicenter $Q^*(G)$ of a group G is the terminal member of the upper quasicentral series $1 = Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_n = Q_{n+1} = Q^*(G)$ of G, where Q_{i+1} is defined by $Q_{i+1}/Q_i = Q(G/Q_i)$. Mukherjee has shown [3, 4] that the quasicenter of a group is nilpotent and the hyperquasicenter is the largest supersolvably immersed subgroup of a group. The proofs of these structure theorems rely on the fact that the powers of QC-elements are again QC-elements.

In this paper we generalize the notion of a quasicentral element in a way which allows the results about the quasicenter and the hyperquasicenter [3, 4] to be extended. All groups mentioned are assumed to be finite.

For a given group G and a fixed prime p, the definition of QC-element might suggest that an element x of G be called a p-quasicentral element provided $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ holds for all p-elements y of G. An apparent difficulty with this definition is that the powers of p-quasicentral elements need not again be p-quasicentral elements. For example, consider the group of order 18 defined by $G = \langle a, b, x \mid a^3 = b^3 = 1 = x^2, [a, b] = 1 = [a, x], [b, x] = a\rangle$. A simple calculation shows that ax is 3-quasicentral while $x = (ax)^3$ is not 3-quasicentral—otherwise $\langle x\rangle\langle b\rangle = \langle b\rangle\langle x\rangle$ shall imply that x normalizes $\langle b\rangle$, which is not the case however. Because of this example we choose to generalize the notion of a QC-element as follows.

DEFINITION 1. Let G be a given group and p a fixed prime. Suppose x is an element of G and let the order of x be written as $|x| = p^r m$ where $(p, m) = 1$. Then x is called a p-quasicentral (p-QC) element of G provided $\langle x^m\rangle\langle y\rangle = \langle y\rangle\langle x^m\rangle$ and $\langle x^{p^r}\rangle\langle y\rangle = \langle y\rangle\langle x^{p^r}\rangle$ hold for all p-elements y of G. (It should be noted that every element of a p'-group is p-QC.)