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EXTENSIONS OF AN INEQUALITY BY PÓLYA AND SCHIFFER FOR VIBRATING MEMBRANES

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The inequality by Pólya and Schiffer considered in this paper is concerned with the sume of the n first reciprocal eigenvalues of the problem $\Delta u + \lambda u = 0$ in G, $u = 0$ on ∂G. First we extend this inequality to the problem of an inhomogeneous membrane $\Delta u + \lambda \rho u = 0$ in G, $u = 0$ on ∂G. Then we prove a sharper form of it for a class of homogeneous membranes with partially free boundary. The proofs are based on a variational characterization for the eigenvalues and use conformal mapping and transplantation arguments.

The inequality by Pólya and Schiffer considered in this paper is concerned with the eigenvalue problem $\Delta\varphi + \lambda\varphi = 0$ in a Jordan domain $G, \varphi = 0$ on ∂G. It can be stated as follows: *Among all domains with given maximal conformal radius $\dot{r}$, the circle yields the minimum of the expression $\sum_{i=1}^{n} \lambda_i^{-1}$.* This theorem is related to the geometrical inequality

$$(1) \qquad \pi\dot{r}^2 \leqq A ,$$

where A denotes the total area of G. The aim of this paper is (i) to extend the inequality by Pólya and Schiffer to the problem of an inhomogeneous membrane fixed on the boundary, (ii) to sharpen it for certain kinds of elastically supported, homogeneous membranes. Instead of considering the problem of an inhomogeneous membrane we will study the equivalent eigenvalue problem $Lu + \lambda u = 0$ where $L = \Delta/\rho$ is the Beltrami operator of an abstract surface with the line element $ds^2 = \rho(dx^2 + dy^2)$. With the help of inequalities by Alexandrow [1], we will derive first some relations between $\dot{r}$, ρ and the Gaussian curvature of the surface. These results will be needed for the theorem concerning the eigenvalue problem. Its proof is essentially based on a method indicated by Hersch in [6] which uses conformal mapping and transplantation arguments. In the last part, we give an isoperimetric inequality for a class of plane membranes. The extremal domain is in this case the circular sector.

1. Geometrical preliminaries.

DEFINITIONS 1.1. Let Σ be an abstract surface given by a Jordan domain G in the z-parameter plane ($z = x + iy$), and by the metric $ds^2 = \rho(z)|dz|^2$ where $\rho(z)$ is an arbitrary positive function in C^2.