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THE REDUCING IDEAL IS A RADICAL

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For any *-algebra $\mathfrak{A}$ the reducing ideal $\mathfrak{A}_R$ of $\mathfrak{A}$ is the intersection of the kernels of all the *-representations of $\mathfrak{A}$. Although the reducing ideal has been called the *-radical, and obviously satisfies $(\mathfrak{A}/\mathfrak{A}_R)_R = \{0\}$, it has not previously been shown to satisfy another of the fundamental properties of an abstract radical except in the case of hermitian Banach *-algebras where it equals the Jacobson radical. In this paper we prove two extension theorems for *-representations. The more important one states that any essential *-representation of a *-ideal of a U^*-algebra (*a fortiori*, of a Banach *-algebra) has a unique extension to a *-representation of the whole algebra. These theorems show in particular that $(\mathfrak{A}_R)_R = \mathfrak{A}_R$ if $\mathfrak{A}$ is either a commutative *-algebra or a U^*-algebra. The somewhat stronger statements which are actually proved, together with previously known properties of the reducing ideal, show that the reducing ideal defines a radical subcategory of each of the following three semi-abelian categories:

(1) Commutative *-algebras and *homomorphisms.

(2) Banach *-algebras and continuous *-homomorphisms.

(3) Banach *-algebras and contractive *-homomorphisms.

The concept of the reducing ideal was introduced by Gelfand and Naimark in their classic paper [2, p. 463]. It has subsequently been studied by Kelley and Vaught [5, p. 51] and the present author [7, p. 63] and [8, p. 930]. The concept is discussed in [10, pp. 210, 226] and [6, p. 259]. In [11, 1479] Yood gave a definition of the *-radical which agrees with our definition for Banach *-algebras but differs for certain other types of *-algebras.

Our main extension theorem (3.1, below) was previously known for B^*-algebras [1, Proposition 2.10.4]. It has a number of applications besides the one discussed here. For example it immediately implies the conclusion of [4, Theorem 23] with hypotheses weaker than those of [4, Theorem 22].

In § 1 we give necessary background information. The case of commutative *-algebras is considered in § 2 and of U^*-algebras in § 3. The category theory results are described in § 4 where we use the terminology of M. Gray [3] for the general theory of radicals.

In general we follow the terminology of Rickart's book [10]. Further details and related results will be found in the author's forthcoming monograph [9].