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A PHRAGMÉN-LINDELÖF THEOREM WITH APPLICATIONS TO $\mathscr{M}(u, v)$ FUNCTIONS

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A well-known theorem of Paley and Wiener asserts that if f is an entire function, its restriction to the real line belongs to the Hilbert space $\mathscr{F}^*L^2(-\tau, \tau)$ (where $\mathscr{F}$ is the Fourier-Plancherel operator) if and only if f is square integrable on the real axis and satisfies $|f(z)| \leqq Ke^{\tau|\mathrm{Im}\, z|}$ for some positive K. The "if" part of this result may be viewed as a Phragmén-Lindelöf type theorem. The pair $(e^{i\tau x}, e^{i\tau x})$ of inner functions can be associated with the above mentioned Hilbert space in a natural way. By replacing this pair by a more general pair (u, v) of inner functions it is possible to define a space $\mathscr{M}(u, v)$ of analytic functions similar to the Paley-Wiener space. For a certain class of inner functions (those of "type $\mathfrak{C}$") it is shown that membership in $\mathscr{M}(u, v)$ is implied by an inequality analogous to the exponential inequality above.

A second application of our results is to star-invariant subspaces of the Hardy space H^2. It is well known that if u is an inner function on the circle and f is in H^2, then in order for f to be in $(uH^2)^\perp$ it is necessary for f to have a meromorphic pseudocontinuation to $|z| > 1$ satisfying

$$|f(z)|^2 \leqq K\frac{1-|u(z)|^2}{1-|z|^2}, \quad |z| > 1 .$$

If u is inner of type $\mathfrak{C}$, it is proved that this necessary condition is also sufficient.

Let $\Gamma = \{e^{i\theta}: 0 < \theta < 2\pi\}$ be the unit circle and

$$R = \{x : -\infty < x < \infty\}$$

the real line considered as point sets in the complex plane $\boldsymbol{C}$. Let D and D_- be the interior and exterior of the unit circle and let Ω and Ω_- be the open upper and open lower half-planes in $\boldsymbol{C}$. A function Φ is *outer* on D or Ω if Φ is holomorphic on D or Ω and of the form

$$\Phi(z) = \exp\int_\Gamma \frac{e^{i\xi}+z}{e^{i\xi}-z}k_1(e^{i\xi})\,\sigma(d\xi), \quad z \in D ,$$

or

$$\Phi(z) = \exp\frac{1}{\pi i}\int_R \frac{1+tz}{t-z}k_2(t)dt, \quad z \in \Omega ,$$