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APPROXIMATE IDENTITIES AND THE STRICT TOPOLOGY

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This paper studies relationships between approximate identities on a B^* algebra A and other properties of the algebra. If A is commutative, conditions on the approximate identity for A are related to topological properties of the spectrum of A. The principal result of this paper is that for a locally compact Hausdorff space S, $C_0(S)$ has an approximate identity that is totally bounded in the strict topology (or compact open topology) if and only if S is paracompact.

1. Introduction. The problem of extending theorems about commutative B^* algebras to the non-commutative case has received a great deal of attention in recent years. Because many proofs made in the commutative case make use of the spectrum (= maximal ideal space), an obvious question is: what is to replace this device in the case of a non-commutative B^* algebra? Various possible replacements have been sought; e.g.; see Akemann [1] and Pedersen [15, 16]. Much progress has been made for certain types of problems by means of restrictions on approximate identities for the algebra in question by Taylor [20, 21], Akemann [2], and others. The class of problems solved or seemingly susceptible to this technique is rather large. This fact and the paucity of results for this class of problems obtained by studying Prim A and the space of equivalence classes of irreducible representations suggest that the approximate identity is a useful tool for extending many commutative theorems to a non-abelian setting. A question that arises immediately in the case of a commutative B^* algebra is: what do restrictions on the approximate identity imply about the spectrum of A and vice versa? Along this line, Collins-Dorroh [6] characterize σ-compactness of the spectrum and ask for necesary and sufficient conditions on S that $C_0(S)$ (in this paper, S always denotes a *locally compact Hausdorff space*) have an approximate identity that is totally bounded in the strict topology (called β by Buck). This paper answers this question and several related ones, including some in the non-commutative context.

2. Preliminaries.

DEFINITION 2.1. Let A be a Banach algebra. An *approximate identity* for A is a net $\{e_\lambda \mid \lambda \in \Lambda\}$ (we generally write simply $\{e_\lambda\}$) with $\lim_\lambda \|e_\lambda x - x\| = \lim_\lambda \|x e_\lambda - x\| = 0$ for $x \in A$ and $\|e_\lambda\| \leqq 1$ for all λ.