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DETERMINING A POLYTOPE BY RADON PARTITIONS

MARILYN BREEN

In an extension of the classical Radon theorem, Hare and Kenelly have introduced the concept of a primitive partition, allowing a reduction to minimal subsets which still possess the necessary intersection property.

Here it is proved that primitive partitions in the vertex set P of a polytope reveal the subsets of P which give rise to faces of conv P, thus determining the combinatorial type of the polytope. Furthermore, the polytope may be reconstructed from various subcollections of the primitive partitions.

2. Preliminary results. Throughout, $|P|$ denotes the cardinality of P. If P is a set of points in R^d, $A \cup B$ is a *Radon partition* for P iff $P = A \cup B$, $A \cap B = \emptyset$, and conv $A \cap$ conv $B \neq \emptyset$. Each of A and B is called half a partition for P and each element of A is said to *oppose* B in the partition. The Radon theorem says that for $P \subseteq R^d$ having at least $d + 2$ points, there exists a Radon partition for P. When P is in general position in R^d and P has exactly $d + 2$ elements, the partition is unique.

In [2], Hare and Kenelly introduce the concept of a primitive partition: For $P \subseteqq R^d$, $A \cup B$ is a Radon partition *in* P iff $A \cup B$ is a Radon partition for a subset S of P. We say that the Radon partition $A \cup B$ *extends* the Radon partition $A' \cup B'$ iff $A' \subseteqq A$ and $B' \subseteqq B$. Finally, $A \cup B$ is called a *primitive partition* in P, or simply a *primitive*, provided it is a Radon partition in P and $A \cup B$ extends the Radon partition $A' \cup B'$ iff $A' = A$ and $B' = B$. It is proved that each Radon partition extends a primitive partition having cardinality at most $d + 2$.

Theorem 1 follows immediately from the results of Hare and Kenelly.

THEOREM 1. *Let P denote a set of $d + 2$ points in R^d and let $A \cup B$ be a primitive for P. Then $|A| + |B| = d + 2$ iff P is in general position.*

COROLLARY 1. *If $A \cup B$ is a primitive for P, $P \subseteqq R^d$, then $A \cup B$ is in general position in R^k for some $k \leqq d$, and $|A| + |B| = k + 2$ for this k.*

THEOREM 2. *If $P \subseteqq R^d$ and $A \cup B$ is a primitive for P, then dim (conv $A \cap$ conv B) $= 0$.*