PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 3, 1972

A COMPLETE COUNTABLE $L^Q_{\omega_1}$ THEORY WITH MAXIMAL MODELS OF MANY CARDINALITIES

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Because of the compactness of first order logic, every structure has a proper elementarily equivalent extension. However, in the countably compact language $L^Q_{\omega_1}$ obtained from first order logic by adding a new quantifier Q and interpreting Qx as "there are at least ω_1 x's such that...," the situation is radically different. Indeed there are structures of countable type which are maximal in the sense of having no proper $L^Q_{\omega_1}$-extensions, and the class S of cardinals admitting such maximal structures is known to be large. Here it is shown that there is a countable complete $L^Q_{\omega_1}$ theory T having maximal models of cardinality κ for each $\kappa \geqq \beth_1$ which is in S. The problem of giving a complete characterization of the maximal model spectra of $L^Q_{\omega_1}$ theories T remains open: what classes of cardinals have the form $\mathrm{Sp}(T) = \{\kappa$: there is a maximal model of T of cardinality $\kappa\}$ for T a (complete, countable) $L^Q_{\omega_1}$ theory.

That S is large is shown in [4]. Assuming the *GCH*, it is particularly simple to describe: S is the set of uncountable cardinals which are less than the first uncountable measurable cardinal and not weakly compact. Here we will need the fact that $\beth_1 \in S$; this is proved in [4] without assuming the *GCH*. The countable compactness of $L^Q_{\omega_1}$ is shown in Fuhrken [2]. For additional results and references on the model theory of $L^Q_{\omega_1}$ see Kiesler [3].

1. Notation and preliminaries.

1.1. *Relatively common notation.* We identify cardinals with initial ordinals, and each ordinal with the set of smaller ordinals. We use α, β, γ for ordinals, κ, λ, μ for cardinals, and m, n for finite cardinals. $S(X) = \{t: t \subseteqq X\}$; cX is the cardinality of X; $\beth_1$ is the cardinality of the continuum; ω_1 the first uncountable cardinal; $\prod_{i \in Y} X_i$ the cartesian product; ${}^Y X$ the set of all functions on Y into X, $f \mid x$ the restriction of the function f to x.

The type $\tau\Sigma$ of a set Σ of formulas is the set of non-logical symbols occurring Σ.

In this paper all structures will be relational structures. Capital german letters are used for structures, and the corresponding roman letters for their universes. Alternatively we may write $|\mathfrak{A}|$ for the universe of $\mathfrak{A}$. The type $\tau\mathfrak{A}$ of $\mathfrak{A}$ is the set of non-logical symbols