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EQUIVARIANT EXTENSIONS OF MAPS

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This paper treats extension and retraction properties in the category $\mathscr{A}_p$ of compact metric spaces with periodic maps of a prime period p; the subspaces and maps in $\mathscr{A}_p$ are called equivariant subspaces and maps, respectively. The motivation of the paper is the following question: Let E be a Euclidean space and $a: E \times E \to E \times E$ be the involution $(x, y) \to (y, x)$, i.e., the symmetry with respect to the diagonal. Suppose that Z is a symmetric (i.e., equivariant) closed subset of $E \times E$ which is an absolute retract; that is, Z is a retract of $E \times E$. When does there exist a symmetric (i.e., equivariant) retraction $E \times E \to Z$?

This is an extension problem in the category $\mathscr{A}_p$. If X and Y are spaces in $\mathscr{A}_p$, A is a closed equivariant subspace of X and $f: A \to Y$ is an equivariant map, then the existence of an extension of f does not, in general, imply the existence of an equivariant extension. It is shown, however, that if A contains all the fixed points of the periodic map and $\dim(X-A) < \infty$, then a condition for the existence of an extension is also sufficient for the existence of an equivariant extension. In particular, it follows that a finite dimensional space X in $\mathscr{A}_p$ is an equivariant ANR (i.e., an absolute neighborhood retract in the category $\mathscr{A}_p$) if and only if it is an ANR and the fixed point set of the periodic map on X is an ANR. Generally speaking, the paper deals with the question of symmetry in extension and retraction problems.

1. Preliminaries.

Suppose that a group G acts on spaces X and Y and that A is an equivariant subspace of X (i.e., A is stable under the action of G). One can then ask for conditions for the extistence of an equivariant extension of f; or for conditions under which the existence of an extension of f implies also the existence of an equivariant extension. A general theorem of this type is due to A. Gleason [6] and R. S. Palais [12, p. 19]:

TIETZE-GLEASON THEOREM. *Let G be an orthogonal group acting on a Euclidean space E by means of orthogonal transformations and let G act on a normal space X. Let A be a closed equivariant subset of X and let $f: A \to E$ be an equivariant map. Then there is an equivariant extension $g: X \to E$ of f.*

This theorem is proved by first extending the map f to some map $\bar{f}: X \to E$ which may not necessarily be equivariant: and then by averaging $\bar{f}$, using a Haar measure on G, to make it equivariant.