PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

ON THE NUMBER OF TYPE-k TRANSLATION-INVARIANT GROUPS

J. HILLEL

The concept of a translation-invariant permutation group was introduced in connection with the problem of constructing "algebras of symmetry-classes of tensors". Such a group is of type-k if it has k orbits. In this paper the number of type-k groups is shown to be the same as the number of divisors of $X^k - 1$ over the two-element field.

Let S_∞ be the group of all permutations of finite degree on the set $\{1, 2, 3, \cdots\}$. If σ is the permutation given by $(a_1b_1)(a_2b_2) \cdots (a_tb_t)$, its *translate* $\sigma^{[1]}$ is defined to be the permutation

$$(a_1 + 1 \;\; b_1 + 1)(a_2 + 1 \;\; b_2 + 1) \cdots (a_t + 1 \;\; b_t + 1) .$$

The definition of the translate of σ is independent of the decomposition of σ into a product of transpositions. A subgroup H of S_∞ is said to be *translation-invariant* (briefly, H is a $t - i$ group) if whenever σ is in H so is $\sigma^{[1]}$.

The translation-invariant groups were first introduced in [1] in connection with the problem of generalizing the construction of the Tensor, Grassmann and Symmetric algebras by using symmetry-classes of tensors (see [2]). The following was proven in [1]: if H is a non-trivial $t - i$ group (assume H moves 1), then the orbits for the action of H on $\{1, 2, 3, \cdots\}$ are $Z_{i,k} = \{i, i + k, i + 2k, \cdots\}, 1 \leqq i \leqq k$, for some $k \geqq 1$. The number of orbits is called the *type* of H. Let $S_{i,\infty}$ (resp. $A_{i,\infty}$) be the group of all (resp. even) permutations on the set $Z_{i,k}$, $1 \leqq i \leqq k$, and let $S_\infty(k) = S_{1,\infty}X \cdots XS_{k,\infty}$, $A_\infty(k) = A_{1,\infty}X \cdots A_{k,\infty}$. For each $k \geqq 1$, these are $t - i$ groups and if H is any type-k $t - i$ group, clearly $H < S_\infty(k)$. Moreover, it was proven that a $t - i$ group contains all the even permutations on each of its orbits, i.e.,

THEOREM 1. *If H is a type-k $t - i$ group then $A_\infty(k) < H < S_\infty(k)$.*

In this presentation we are concerned with determining the number of type-k $t - i$ groups for each $k \geqq 1$. In [1] it was proven that:

THEOREM 2. *There are $2^n + 1$ $t - i$ groups of type-2^n, $n \geqq 0$.*

The above theorem was proved by looking at some special features of the lattice of the type-k $t - i$ groups. However, here we will show that the number of type-k $t - i$ groups is the same as the number of factors of the polynomial $X^k - 1$ over the two-element field F_2 and