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ESSENTIAL PRODUCTS OF NONSINGULAR RINGS

K. R. GOODEARL

By an essential product of two rings is meant a subdirect product which contains an essential right ideal of the direct product. The aim of this paper is to investigate the utility of this concept in the study of nonsingular rings. The first section derives some basic properties of essential products and develops some criteria for recognizing essential products. In the second section, a study of the socles of nonsingular modules leads to a theorem that any nonsingular ring is an essential product of a ring with essential socle and a ring with zero socle. The third section is devoted to a theorem which tells when an essential product can be a splitting ring, i.e., a ring such that the singular submodule of any right module is a direct summand. In the final section, this theorem is used to construct two examples of splitting rings of types previously unknown.

In this paper all rings are associative with identity, and all modules are unital. We also require that a subring of a ring have the same identity as the ring. Unless otherwise noted, all modules are right modules.

Inasmuch as we use singular and nonsingular modules throughout this paper, we recall the relevant definitions here. Given a ring R, we use $\mathscr{S}(R)$ to denote the collection of essential right ideals of R; then the *singular submodule* of a right R-module A is the set $Z_R(A) = \{a \in A \mid aI = 0 \text{ for some } I \in \mathscr{S}(R)\}$. The module A is said to be *singular* [*nonsingular*] provided $Z_R(A) = A[Z_R(A) = 0]$. The singular submodule of R_R is a two-sided ideal of R, called the *right singular ideal* of R and denoted $Z_r(R)$; R is a *right nonsingular ring* when $Z_r(R) = 0$.

1. Essential products. Given two right nonsingular rings R_1 and R_2, we define an *essential product* of R_1 and R_2 to be any subdirect product R of R_1 and R_2 which contains an essential right ideal of $R_1 \times R_2$. [Recall that for R to be a subdirect product of R_1 and R_2, R must be a subring of $R_1 \times R_2$ such that the projections $R \to R_1$ and $R \to R_2$ are both surjective.] The aim of this section is to consider the relationships among singular and nonsingular modules over R_1, R_2, and R, and to establish criteria for judging which rings are essential products.

N.B.-For the first three propositions in this section, we assume that R is an essential product of two right nonsingular rings R_1 and