PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

MAXIMAL SUBFIELDS OF TENSOR PRODUCTS

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Let D_1 and D_2 be finite-dimensional division rings with center K such that $D_1 \otimes_K D_2$ is a division ring. If L_1 and L_2 are maximal subfields of D_1 and D_2, respectively, then clearly $L_1 \otimes_K L_2$ is a maximal subfield of $D_1 \otimes_K D_2$. In this note the converse question is considered: does there exist a maximal subfield L of $D_1 \otimes_K D_2$ which is not isomorphic to $L_1 \otimes_K L_2$ for maximal subfields L_1 and L_2 of D_1 and D_2? Examples are given to show that such noncomposite L may fail to exist even when K is a local field. For K an algebraic number field, however, it is shown that infinitely many noncomposite L always exist.

We say that a division algebra with center a field K is a *K-division ring* if it is finite-dimensional over K. Throughout this note D_1 and D_2 will denote K-division rings such that $D_1 \otimes_K D_2$ is a K-division ring. We say that a maximal subfield L of $D_1 \otimes_K D_2$ is a *composite* if $L \cong L_1 \otimes_K L_2$ where L_1 and L_2 are maximal subfields of D_1 and D_2, respectively.

A sufficient condition for $D_1 \otimes_K D_2$ to be a division ring is for $([D_1: K], [D_2: K]) = 1$ [2, Theorem 10, p. 52]. This condition is necessary if K is either an algebraic number field or a local field since for these K the exponent of a K-division ring equals its index [2, Theorem 25, p. 144, and Theorem 32, p. 149]. This condition is not, however, necessary for K arbitrary, as is shown in [1]. We begin by determining, for the case when $([D_1: K], [D_2: K]) = 1$ necessary and sufficient conditions for a maximal subfield of $D_1 \otimes_K D_2$ to be a composite.

THEOREM 1. *Let D_1 and D_2 be K-division rings such that $([D_1: K], [D_2: K]) = 1$, and let L be a maximal subfield of $D_1 \otimes_K D_2$. Then L is a composite if and only if L has subfields L_1 and L_2 with $[L_1: K]^2 = [D_1: K]$ and $[L_2: K]^2 = [D_2: K]$.*

Proof. Let $n_i = [D_i: K]^{1/2}$, $i = 1, 2$. If L_i is a maximal subfield of D_i then $[L_i: K] = n_i$, $i = 1, 2$. It follows that if $L = L_1 \otimes_K L_2$ is a composite with L_i a maximal subfield of D_i, then $[L_i: K] = n_i$, $i = 1, 2$. This establishes one direction of the Theorem.

Suppose now that L has subfields L_1 and L_2 with $[L_i: K] = n_i$, $i = 1, 2$. Since L is a maximal subfield of $D_1 \otimes_K D_2$ we have $[L: K] = n_1 n_2$. As $(n_1, n_2) = 1$, it follows that $L \cong L_1 \otimes_K L_2$. Thus to conclude L is a composite we need only show that L_i splits D_i,