PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 1, 1973

THE MULTIDIMENSIONAL CONTENT OF THE FRUSTUM OF THE SIMPLEX

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The content of the intersection of a simplex with a semispace is computed by means of a dissection technique. An efficient algorithm, suitable for automatic calculation, is given. For an n-dimensional space, the algorithm needs only $n-1$ storage location at most, and requires $\sim n^2$ operations.

Introduction. The simplex S considered is the $(n-1)$-dimensional polytope defined as the convex hull of its n vertices V_j in R^n, the Euclidean space in n dimensions:

$$V_j \equiv \{v_{j1}, v_{j2}, \cdots, v_{jn}\} ,$$

where

$$v_{ji} = \delta_{ji} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \qquad j, i = 1, 2, \cdots, n .$$

This choice of geometry is convenient in certain applications of the algorithm that arise in statistical mechanics and allocation theory, but does not result in a loss of generality (see Appendix).

The frustum F_1 of S is defined as the nonempty intersection of S with the semispace σ:

$$\sigma \equiv \{x_1, x_2, \cdots, x_n \mid \sum p_i x_i \leqq G\} ,$$

where the real numbers p_i are the coefficients that characterize the hyperplane γ, boundary of the semispace.

With a minor loss of generality, that will be removed subsequently, it will be assumed that

$$(1) \qquad \begin{aligned} &p_1 < p_2 < \cdots < p_n . \\ &p_i \neq G \qquad\qquad i = 1, 2, \cdots, n . \end{aligned}$$

It can then be immediately verified that the condition for $S \cap \sigma$ to be nonempty is

$$p_1 \leqq G \leqq p_n .$$

The content $C[F_1]$ of the frustum can be represented as

$$\iint \cdots \int_X dV^{n-1}$$

where the region X is defined by the following constraints: