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ON THE FUNDAMENTAL UNIT OF A PURELY CUBIC FIELD

RONALD J. RUDMAN

Let $a = D^3 + d$, where a, D, d are rational integers with $D, a > 0$, $|d| > 1$, and $d \mid 3D^2$. It is proved that the fundamental unit of the field $Q(\omega)$, where $\omega = \sqrt[3]{a}$, is $(\omega - D)^3/d$ with only six exceptions.

1. Introduction. The purpose of this paper is to establish the following result:

THEOREM 1. *Let $a = D^3 + d$, where $a, D, d \in Z$, with $a, D > 0$, $|d| > 1$, and a cubefree. Then $\varepsilon = (\omega - D)^3/d$, where $\omega = \sqrt[3]{a}$, is a unit of $K = Q(\omega)$ if and only if $d \mid 3D^2$. Moreover, in this case $\varepsilon = \eta$, the fundamental unit of K, except for $(D, d) = (2, -6)$, $(1, 3)$, $(2, 2)$, $(3, 1)$, and $(5, -25)$, where $\varepsilon = \eta^2$, and $(2, -4)$, where $\varepsilon = \eta^3$.*

Here, Z, Q denote respectively the rational integers and the field of rationals.

Theorem 1 is an extension of a result of Stender [4], who showed that when

(1) $\quad a = D^3 + d\,, \qquad d \mid D,\ d > 1$

(2) $\quad a = D^3 + 3d\,, \qquad d \mid D,\ 3d \leqq D,\ d > 0$

(3) $\quad a = D^3 + 3D\,, \qquad D \geqq 2\,,$

(4) $\quad a = D^3 - d\,, \qquad d \mid D,\ 4 < 4d \leqq D\,,$

or

(5) $\quad a = D^3 - 3d\,, \qquad d \mid D,\ 12d \leqq D,\ d > 0$

$\varepsilon = (\omega - D)^3/(\omega^3 - D^3) = \eta$, except for $(D, d) = (2, 2)$ in (1), where $e = \eta^2$. The case $d = 1$ in (1) and (4) had already been settled by Nagell [2], who proved that $\varepsilon = \eta$ with the single exception of $a = 28$, when $\varepsilon = \eta^2$. The method of proof used here follows [4].

2. Preliminaries. We now make the assumption that $d \mid 3D^2$.

Since a is cubefree we put $a = mn^2$ with m squarefree. Also, d is cubefree, as $d \mid 3a$.

Let $\bar{a} = m^2n$, $\bar{\omega} = \sqrt[3]{a}$, and ζ be the fundamental unit of the ring $R = [1, \omega, \bar{\omega}]$. It is well known that if $a \not\equiv \pm 1 \pmod 9$, an integral basis for K is $\langle 1, \omega, \bar{\omega}\rangle$ (a field of the first kind). However, if $a \equiv \pm 1 \pmod 9$, an integral basis for K is given by