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THE COHOMOLOGICAL DESCRIPTION OF A TORUS ACTION

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The theorem proved in this paper is an example of a "regularity" theorem in the study of topological group actions—that is, it shows that a general topological action of a group continues to have certain properties of "linear" actions. Consider an action of a torus T on a cohomology n-sphere X, with fixed point set the cohomology r-sphere F. Consider the map $H^n(X_T; Z) \to H^n(F_T; Z)$, and let $c\eta$ be the image of the generator of $H^n(X; Z)$, considered as lying in $H^{n-r}(BT; Z)$, where c is an integer and η has no nontrivial integer divisors. The polynomial part η is well understood. The theorem will evaluate the integer part c in the following sense: in the linear case, c can be easily expressed in terms of the dimensions of the fixed point sets of various nonconnected subgroups of T. It is shown that this formula continues to hold in the general topological case, given some weak assumptions. There is also a corresponding result for the case $F = \varnothing$.

The main tool will be the fibration $\pi: X_T \to B_T \equiv BT$, where X_T is as usual $E_T \times {}_T X$. We will use the usual limit arguments to allow ourselves to pretend that E_T is compact. Cohomology will be sheaf cohomology with compact supports (which will not usually be indicated). The spectral sequence of $X_T \to BT$ with coefficients in A will be denoted $E_r(X_T; A)$. The fixed point set of T acting on X will be denoted $F(T, X) \equiv F(T)$. $X \sim {}_Z Y (X \sim {}_p Y)$ will mean that X is a compact Z-cohomology (Z_p-cohomology) manifold with $Z(Z_p)$ cohomology ring the same as that of Y. $\dim_p(X)$ or $\dim_Z(X)$ will be the usual cohomological dimension of X over Z_p or Z. See [1] or [2] for details. For an abelian group A, let $\mathscr{F}A$ be $A/\text{Torsion}(A)$.

If a torus T acts on a space X, a subtorus H of T is said to be *distinguished* if $F(H) \supsetneqq F(K)$ for any subtorus K which has $K \supsetneqq H$. In particular, the distinguished corank one subtori of T are those subtori H of corank one in T that have $F(H) \supsetneqq F(T)$. Recall that given a corank one subtorus of T, there is a corresponding integer-valued linear functional on the Lie algebra of T, a corresponding element of $H^1(T; Z)$ and a corresponding element (not divisible by any integer) in $H^2(BT; Z)$.

Now consider a torus T acting on $X \sim {}_Z S^n$. Let $F(T) \sim {}_Z S^r$, and look at $F_T \subseteqq X_T$. Consider the cases $r > 0$, $r = 0$, and $r = -1$ $(F(T) = \varnothing)$ separately.