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LENGTH OF PERIOD OF SIMPLE CONTINUED FRACTION EXPANSION OF $\sqrt{d}$

DEAN R. HICKERSON

In this article, the length, $p(d)$, of the period of the simple continued fraction (s.c.f.) for $\sqrt{d}$ is discussed, where d is a positive integer, not a perfect square. In particular, it is shown that

$$p(d) < d^{1/2+\log 2/\log\log d+O(\log\log\log d/(\log\log d)^2)} .$$

In addition, some properties of the complete quotients of the s.c.f. expansion of $\sqrt{d}$ are developed.

It is well known that the s.c.f. expansion for $\sqrt{d}$ is periodic if d is a positive integer, not a perfect square. Throughout this paper, $p(d)$ will denote the length of this period. It is shown in [2] (page 294), that $p(d) < 2d$. Computer calculation of $p(d)$ originally suggested that $p(d) \leqq 2[\sqrt{d}]$. This was shown to be false for $d = 1726$, for which $p(d) = 88$ and $2[\sqrt{d}] = 82$. Further calculation revealed 3 more counterexamples for $d \leqq 3000$. They were $p(2011) = 94$ while $2[\sqrt{2011}] = 88$, $p(2566) = 102$ while $2[\sqrt{2566}] = 100$, and $p(2671) = 104$ while $2[\sqrt{2671}] = 102$.

This suggests as a conjecture that

$$p(d) = O(d^{1/2}) \quad \text{and} \quad p(d) \neq o(d^{1/2}) .$$

It follows from the corollary to Theorem 2 that

$$p(d) = O(d^{1/2+\varepsilon})$$

or more precisely, that

$$p(d) < d^{1/2+\log 2/\log\log d+O(\log\log\log d/(\log\log d)^2)} .$$

We will need the following results which are given in or follow from §§ 7.1–7.4 and 7.7 of [1].

(1) Any periodic s.c.f. is a quadratic irrational number, and conversely.

(2) The s.c.f. expansion of the real quadratic irrational number $(a+\sqrt{b})/c$ is purely periodic if and only if $(a+\sqrt{b})/c > 1$ and $-1 < (a-\sqrt{b})/c < 0$.

(3) Any quadratic irrational number ξ_0 may be put in the form $\xi_0 = (m_0+\sqrt{d})/q_0$, where d, m_0, and q_0 are integers, $q_0 \neq 0$, $d \geqq 1$, d is not a perfect square, and $q_0 \mid (d - m_0^2)$. We may then define infinite