PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 2, 1973

RADIAL AVERAGING TRANSFORMATIONS WITH VARIOUS METRICS

CATHERINE BANDLE AND MOSHE MARCUS

In [5] one of the authors introduced the notion of a radial averaging transformation of domains in the plane, which was based on the metric given by the line element $ds^2 = (1/r^2)(dx_1^2 + dx_2^2)$ where (x_1, x_2) are the cartesian and (r, θ) are the polar coordinates. This transformation is useful in obtaining estimates for conformal capacity of condensers and for conformal radius of domains. In this paper we discuss averaging transformations in m-dimensional spaces ($m \geqq 2$), based on various metrics of the form $ds^2 = g^2(r) \sum_{i=1}^{m} (dx_i)^2$, where $g(r)$ is a positive, continuous function of $r (0 < r < \infty)$.

With the help of these transformations we are able to obtain estimates for energy integrals of the form

$$\int_{\Omega} |\nabla F|^2 g(r) r^{3-m} dx \qquad (dx = dx_1 dx_2 \cdots dx_m)\,.$$

These estimates can be used to compare capacities of different condensers filled with nonhomogeneous dielectric [cf. Kühnau [3] and the literature cited there]. As a further application we derive inequalities for conformal capacity and conformal radius in the plane and similar results in higher dimensional spaces. In this direction we have results for the case where $g(r) = r^\beta \ \beta \geqq m - 3$. They include the symmetrization results obtained by Szegö in [7]. The method presented seems to be quite general, and we believe that it might be employed also with other classes of metrics g.

1. Estimates for energy integrals. Let $g(r)$ be a positive continuous function for $0 < r < \infty$ and let $G(r)$ be a primitive of g. $(r, \theta_1, \cdots, \theta_{m-1})$ are the polar coordinates defined in the following way:

$$x_1 = r\cos\theta_1,\ x_2 = r\sin\theta_1\cos\theta_2,\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3,$$
$$\cdots\cdots$$
$$x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-2}\sin\theta_{m-1}$$

where $0 \leqq \theta_i \leqq \pi$ for $i = 1, \cdots, m-2$ and $-\pi \leqq \theta_{m-1} \leqq \pi$. Let ρ be a fixed positive number and set:

$$\begin{cases} u = G(r) - G(\rho) \\ v_i = \theta_i \qquad\qquad i = 1, \cdots, m-1\,. \end{cases} \tag{1.1}$$

Let Ω be an open set in R^m which does not contain the sphere $\{x; |x| \leqq \rho\}$ and the hyperplanes $\theta_i = 0 \ i = 1, 2, \cdots, n-2$. Then for $F(x) \in C^1(\Omega)$ $[x = (x_1, x_2, \cdots, x_m)]$ we have