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GEOMETRIC PROPERTIES OF SOBOLEV MAPPINGS

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If f is a mapping from an open k-cube in R^k into R^n, $2 \leqq k \leqq n$, whose coordinate functions belong to appropriate Sobolev spaces, then the area of f is the integral with respect to k dimensional Hausdorff measure over R^n of a nonnegative integer valued multiplicity function.

1. Introduction. If $f: Q \to R^n$, Q an open k-cube in R^k, $2 \leqq k \leqq n$, is a mapping whose coordinate functions belong to appropriate Sobolev classes, it was shown in [6] that f is $k-1$ continuous and that the area of f, as defined in [5], is equal to the classical Jacobian integral. The purpose of this paper is to investigate, using the theory of currents as in [2], the geometric-measure theoretic properties of such a surface and to show that the area is equal to the integral with respect to k dimensional Hausdorff measure in R^n of an integer valued multiplicity function.

2. Suppose k and n are integers with $2 \leqq k \leqq n$. Let

$$Q = R^k \cap \{x\colon 0 < x_i < 1 \text{ for } 1 \leqq i \leqq k\}$$

and let $\Lambda(k, n)$ denote the set of all k-tuples $\lambda = (\lambda_1, \cdots, \lambda_k)$ of integers such that $1 \leqq \lambda_1 < \cdots < \lambda_k \leqq n$. Suppose $f: Q \to R^n$, $f = (f^1, \cdots, f^n)$, $f^i \in W^1_{p_i}(Q)$, $p_i > k-1$, and $\sum_{j=1}^k 1/p_{\lambda_j} \leqq 1$ whenever $\lambda \in \Lambda(k, n)$. Here $W^1_p(Q)$ denotes those functions in $L^p(Q)$ whose distribution partial derivatives are functions in $L^p(Q)$.

Let $e_1, \cdots, e_n$ be the usual basis for R^n and let

$$e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_k} ,$$

$\lambda \in \Lambda(k, n)$, denote the corresponding basis for the space of k-vectors in R^n. For $\lambda \in \Lambda(k, n)$ let p^λ denote the orthogonal projection of R^n onto R^k defined by letting

$$p^\lambda(y) = (y_{\lambda_1}, \cdots, y_{\lambda_k}) \quad \text{for} \quad y = (y_1, \cdots, y_n) \in R^n .$$

For almost every (in the sense of k dimensional Lebesgue measure $\mathscr{L}_k$) $x \in Q$, let $Jf(x) = \sum_{\lambda \in \Lambda(k,n)} Jf^\lambda(x) e_\lambda$ where Jf^λ denotes the determinant of the matrix of distribution partial derivatives of $f^\lambda = p^\lambda \circ f$. In [6] it was shown that the area of f, as defined in [5] is equal to $\int_Q |Jf(x)|\,dx$ where $|Jf(x)|$ is the Euclidean norm of the k-vector $Jf(x)$.