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AN ERGODIC PROPERTY OF LOCALLY COMPACT AMENABLE SEMIGROUPS

JAMES C. S. WONG

Let $M(S)$ be the Banach algebra of all bounded regular Borel measures on a locally compact semigroup S with variation norm and convolution as multiplication and $M_0(S)$ the probability measures in $M(S)$. We obtain necessary and sufficient conditions for the semigroup S to have the (ergodic) property that for each $\nu \in M(S)$, $|\nu(S)| = \inf\{\|\nu*\mu\|: \mu \in M_0(S)\}$, an extension of a known result for locally compact groups.

1. **Notations and terminologies.** We shall follow Hewitt and Ross [9] for basic notations and terminologies concerning integration on locally compact spaces. Let S be a locally compact semigroup with jointly continuous multiplication and $M(S)$ the Banach algebra of all bounded regular Borel measures on S with total variation norm and convolution $\mu*\nu$, μ, $\nu \in M(S)$ as multiplication where

$$\int f d\mu*\nu = \iint f(xy)d\mu(x)d\nu(y) = \iint f(xy)d\nu(y)d\mu(x)$$

for $f \in C_0(S)$ the space of all continuous functions on S which vanish at infinity. (See for example [1], [6], or [18].) Let $M_0(S) = \{\mu \in M(S): \mu \geqq 0 \text{ and } \|\mu\| = 1\}$ be the set of all probability measures in $M(S)$. Consider the continuous dual $M(S)^*$ of $M(S)$. Denote by 1 the element in $M(S)^*$ such that $1(\mu) = \int d\mu = \mu(S)$, $\mu \in M(S)$. Clearly $\|1\| = 1$.

2. **Convolution of functionals and measures, means.** Let $F \in M(S)^*$, $\mu \in M(S)$, we define a linear functional $l_\mu F = \mu \odot F$ on $M(S)$ by $\mu \odot F(\nu) = F(\mu*\nu)$, $\nu \in M(S)$. Clearly $\mu \odot F \in M(S)^*$. In fact $\|\mu \odot F\| \leqq \|\mu\| \cdot \|F\|$. Similarly we define $F \odot \mu = r_\mu F$.

A linear functional $M \in M(S)^{**}$ is called a mean if $M(F) \geqq 0$ if $F \geqq 0$ and $M(1) = 1$. Here $F \geqq 0$ means $F(\mu) \geqq 0$ for all $\mu \geqq 0$ in $M(S)$. An equivalent definition is

$$\inf\{F(\mu): \mu \in M_0(S)\} \leqq M(F) \leqq \sup\{F(\mu): \mu \in M_0(S)\}$$

for any $F \in M(S)^*$.

Consequently $\|M\| = M(1) = 1$ for any mean M on $M(S)^*$. It follows that the set of means is weak* compact and convex. Each probability measure $\mu \in M_0(S)$ is a mean if we put $\mu(F) = F(\mu)$, $F \in$