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AUTOMORPHISMS OF EXTRA SPECIAL GROUPS AND NONVANISHING DEGREE 2 COHOMOLOGY

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If E is an extra-special 2-group, it is known that Aut $(E)/$ Inn (E) is isomorphic to an orthogonal group. We prove that this extension is nonsplit, except in small cases. As a consequence, the nonvanishing of the second cohomology groups of certain classical groups (defined over F_2) on their standard modules may be inferred. Also, a criterion for a subgroup of these orthogonal groups to have a nonsplit extension over the standard module is given.

1. Introduction. Let E be an extra-special 2-group of order 2^{2n+1}, $n \geqq 1$. That is, $E' = Z(E)$ and E/E' is elementary abelian. Any extra-special group may be expressed as a central product of dihedral groups D_8 of order 8 and quaternion groups Q_8 of order 8, with the central subgroup of order 2 in each factor amalgamated. The expression of E as such a central product is not unique in general because $D_8 \circ D_8 \cong Q_8 \circ Q_8$. However, the number of quaternion central factors is unique modulo 2 for any such expression (see [9] or [12]).

The commutator quotient E/E' may be regarded as a vector space over the field of two elements F_2 equipped with a quadratic form q, where $q(xE') = x^2 \in E'$, for $x \in E$ (we identify E' with the additive group of F_2). The bilinear form b associated with q is defined by $b(xE', yE') = q(xE')q(yE')q(xyE')$ (in multiplicative notation, and, in fact $b(xE', yE') = x^{-1}y^{-1}xy = [x, y]$ is the commutator of x and y. Clearly, any automorphism of E induces an automorphism of E/E' which preserves this quadratic form. Hence, as Inn (E) consider with the group of central automorphisms of E, Aut $(E)/$Inn (E) is isomorphic with a subgroup of some orthogonal group $0^{\pm}(2n, 2)$.

On the other hand, it is not difficult to see that of the full orthogonal group may be lifted to automorphisms of E ([12], 13.9). However, as we shall prove, there is usually no subgroup of Aut (E), (Aut $(E)'$) isomorphic to the relevant (simple) orthogonal group complementing Inn (E). For the case $|E| \geqq 2^9$, the argument is surprisingly easy, and gives a criterion for a subgroups of $0^{\epsilon}(2n, 2)$ to have a nonsplit extension over the standard $2n$-dimensional module.

With similar considerations, one can see that, if E is an extra-special 2-group of order 2^{2n+1}, $Y \cong Z_4$, the group $E \circ Y$ (with a group of order two amalgamated) has $Z_2 \times \mathrm{Sp}(2n, 2)$ as the outer automorphism group (the isomorphism $D_8 \circ Z_4 \cong Q_8 \circ Z_4$ is useful here; [12],