PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 1, 1973

CROSS-SECTIONS OF DECOMPOSITIONS

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The following question was raised by R. H. Bing: "Is it true that if G is a monotone decomposition of E^3 into straight line intervals and one-point sets, then E^3/G is homeomorphic to E^3?" In his paper "Point-like decompositions of E^3" he described a possible counter example. This example has the interesting property that it has many tame cross-sections, but if its decomposition space is homeomorphic to E^3, its set of nondegenerate elements would have to form a wild Cantor set. This suggests that it would be interesting to study the connection between the embedding of a cross-section and the embedding of the set of nondegenerate elements in the decomposition space.

1. Introduction. Most of the terminology and notation used in this paper is standard. The reader is referred to [1], [3], [4], and [6].

If S is a 2-sphere in E^3, then by Int S we will mean the bounded component of E^3-S and by Ext S, the unbounded component.

Let G be an upper semi-continuous decomposition of E^3 and let H be the set of all nondegenerate elements of G. We will say that a set $R \subset E^3$ is a cross-section of G if (i) $R \cap h$ is a singleton for each $h \in H$, and (ii) the natural map P restricted to R is homeomorphism onto $\overline{P(H)}$. We note that cross-sections exist only for certain decompositions. A simple example may be constructed as follows: Let $a_n = 1/n$, for $n = 1, 2, \cdots$ and let $b_n = -1/n$ for $n = 1, 2, \cdots$. Let the set of nondegenerate elements of our decomposition consist of the closed interval from $(0, 1, 0)$ to $(0, -1, 0)$, the closed interval from $(a_n, 1/2, 0)$ to $(a_n, 1, 0)$ for each positive integer n, and the closed interval from $(b_n, -1/2, 0)$ to $(b_n, -1, 0)$ for each positive integer n.

II. Cross-sections of decompositions. The following question naturally arises: How are the embeddings of a cross-section R and $\overline{P(H)}$ related when E^3/G is homeomorphic to E^3? We will give some partial results to this question.

THEOREM 1. *Let G be an upper semi-continuous decomposition of E^3 into points and straight line intervals pointing in only a countable number of directions whose lengths are bounded away from zero such that $P(H)$ is a compact 0-dimensional set. If there exists a cross-section C of G then C is tame.*

Proof. In the special case where the elements of H point in only