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HOMOMORPHISMS OF MATRIX RINGS INTO MATRIX RINGS

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Let $V_n(R_n)$ be the universal ring with respect to embeddings of the matrix ring R_n into $n \times n$ matrix rings over commutative rings. A construction and a representation is given for this ring. As a main tool in the construction, it is proved that every R homomorphism of R_n, R a commutative ring, is the restriction of an inner automorphism of U_n, for some $U \supseteq R$. Using this, a necessary and sufficient condition for n^2 matrices in R_n to be matrix units is given.

1. Introduction and notations. All rings to be considered in this paper, except those denoted specifically as matrix rings, will be commutative rings with unit. All homomorphisms are unitary. The unit of a subring coincides with the unit of its over-ring.

Denote by R_n the ring of $n \times n$ matrices over a ring R. Let $\eta: R \rightarrow S$ be a ring homomorphism then η induces a homomorphism $\eta_n: R_n \rightarrow S_n$ given by: $\eta_n(r_{ij}) = (\eta(r_{ij}))$. If $A \in R_n$, $(A)_{ij}$ will denote the (i, j)th entry of A. The identity element and the standard matrix units of all matrix rings will be denoted by I and $\{E_{ij}\}$ respectively.

Let A be an R algebra. It was proved by Amitsur ([1], Theorem 2) that there exists a commutative R algebra $V_m^R(A)$, and a map $\rho: A \rightarrow (V_m^R(A))_m$ which is universal for homomorphisms of A into $m \times m$ matrix rings over commutative rings, i.e.;

(1) For every $\tau: A \rightarrow H_m$, with H a commutative R algebra, there exists a homomorphism η: $V_m^R(A) \rightarrow H$ such that the following diagram is commutative;

$$\begin{array}{ccc} A & \xrightarrow{\rho} & (V_m^R(A))_m \\ & \searrow^{\tau} & \downarrow \eta_m \\ & & H_m \end{array}$$

(2) $V_m^R(A)$ is generated over R by the entries $\{[\rho(a)]_{ij} \mid a \in A\}$.

Properties (1) and (2) determine $V_m^R(A)$ up to isomorphism and ρ up to a multiple by an isomorphism of $V_m^R(A)$.

In this paper we will give an explicit construction for the ring $V_m^R(R_n)$. The case $n = m$ will be treated separately. We start with investigating the nature of R-homomorphisms of R_n into itself.