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EQUALLY PARTITIONED GROUPS

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It is proved that the only finite groups which can be partitioned by subgroups of equal orders are the p-groups of exponent p. The connection between equally partitioned groups and Sperner spaces is discussed. It is also proved that finite groups partitioned by pairwise permutable subgroups are abelian.

1. Let G be a group and let Π be a collection of proper subgroups of G. Then Π is said to *partition* G if every nonidentity element of G is contained in exactly one $H \in \Pi$. If G is a p-group of exponent p and $|G| > p$, we may let Π be the set of cyclic subgroups of G. Then Π is a partition consisting of subgroups of equal finite orders. Our main result is that the p-groups of exponent p are the only finite groups which can be equally partitioned.

The methods of proof in this paper depend strongly on the finiteness of the group and give no information about which infinite groups can be partitioned by subgroups of equal finite orders.

I began to consider equally partitioned groups after attending a lecture by Prof. A. Barlotti on Sperner spaces. Examples of these geometric objects (which generalize affine spaces) are provided by such groups. In fact the Sperner spaces which arise from finite equally partitioned groups are exactly those which Barlotti and Cofman [2] call translation spaces. This will be discussed further in § 3.

2. Only finite groups will be considered. A great deal is known about partitioned groups. (We mention in particular the papers [1] and [5].) Our theorem, however, is much more elementary and does not depend on the deeper results.

The following easy lemma (which appears in [1]) is crucial to the study of partitioned groups.

LEMMA 1. *Let G be partitioned by Π and let $x, y \in G - \{1\}$ with $xy = yx$. Suppose x and y lie in different elements of Π. Then x and y have equal prime orders.*

Proof. Suppose $o(x) < o(y)$. Then $(xy)^{o(x)} = y^{o(x)} \neq 1$. Let $y \in H \in \Pi$ then $(xy)^{o(x)} \in H$ and hence $xy \in H$. Thus $x \in H$, a contradiction. Therefore $o(x) = o(y)$. Similarly, $o(x^n) = o(y) = o(x)$ for positive integers $n < o(x)$. It follows that $o(x)$ is prime.

LEMMA 2. *Let G be equally partitioned by Π and let $X \subseteqq G$ be*