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ON NUMERICAL RANGES OF ELEMENTS OF LOCALLY m-CONVEX ALGEBRAS

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The concept of numerical range is extended from normed algebras to locally m-convex algebras. It is shown that the approximating relations between the numerical range and the spectrum of an element are preserved in the generalization. The set of elements with bounded numerical range is characterized and the relation between boundedness of the spectrum and of the numerical range is discussed. The Vidav-Palmer theory is generalized to give a characterization of b^*-algebras by numerical range.

In a complex unital Banach algebra the numerical range of an element is a set of complex numbers which can be used to approximate the spectrum of an element. In a complex locally m-convex algebra with identity, for each element we define a set of numerical ranges and establish similar approximation to the spectrum of the element. In a normed algebra the spectrum and the numerical range of each element are bounded sets, but in a locally m-convex algebra the spectrum and the numerical ranges of an element may be unbounded. For a locally m-convex algebra with identity we characterize those elements with a bounded numerical range as an important normed subalgebra, and we discuss the relation between boundedness of the spectrum and the numerical ranges. In the normed algebra theory the study of hermitian elements, those with real numerical range, has led to the important Vidav-Palmer theory characterizing unital B^*-algebras among unital Banach algebras. We generalize the results of this theory to a characterization of b^*-algebras by numerical range.

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1. The numerical ranges of an element. For a complex normed algebra $(A, ||\cdot||)$ with identity 1 where $||1|| = 1$, i.e., a *complex unital normed algebra,* we define the set

$$D(A, ||\cdot||; 1) \equiv \{f \in A' : f(1) = 1 \quad \text{and} \quad ||f|| = 1\} .$$

For each $a \in A$ we define the *numerical range of* a as the set

$$V(A, ||\cdot||; a) \equiv \{f(a) : f \in D(A, ||\cdot||; 1)\} ,$$