PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

ON THE LATTICE OF PROXIMITIES OF ČECH COMPATIBLE WITH A GIVEN CLOSURE SPACE

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Let (X, c) be a Čech closure space. By $\mathfrak{M}$ we denote the family of all proximities of Čech on X which induce c. $\mathfrak{M}$ is known to be a complete lattice under set inclusion as ordering. The analogue of the R_0 separation axiom as defined for topological spaces is introduced into closure spaces. R_0-closure spaces are exactly those spaces for which $\mathfrak{M} \neq \phi$. Other characterizations for R_0-closure spaces are presented. The most interesting one is: every R_0-closure space is a subspace of a product of a certain number of copies of a fixed R_0-closure space. A number of techniques for constructing elements of $\mathfrak{M}$ are developed. By means of one of these constructions, all covers of any member of $\mathfrak{M}$ can be obtained. Using these constructions the following structural properties of $\mathfrak{M}$ are derived: $\mathfrak{M}$ is strongly atomic, $\mathfrak{M}$ is distributive, $\mathfrak{M}$ has no antiatoms, $|\mathfrak{M}| = 0, 1$ or $|\mathfrak{M}| \geqq 2^{2^{\aleph_0}}$.

1. Introduction. E. Čech in [2] has studied a basic proximity structure (see Definition 1.3). The closure operator induced by such a structure is in general not a Kuratowski closure operator, since it may fail to satisfy the condition $c(c(A)) \subset c(A)$, however it satisfies the other three conditions and thus (X, c) is a closure space (Definition 1.1). Since Čech called his basic proximity just a "proximity" and since this term is commonly used to denote a proximity of Efremovič, we shall refer to the basic proximities of Čech as $\check{C}$-proximities. We did not wish to use the name "Čech proximity" because this term already has another meaning in the literature [2, p. 447].

This paper is primarily concerned with a study of the order structure of the family $\mathfrak{M}$ of all $\check{C}$-proximities which induce the same closure operator on a given set. Čech [2] proved that $\mathfrak{M}$ is a complete lattice. He characterized least upper bounds in $\mathfrak{M}$, the least and greatest elements in $\mathfrak{M}$, and those closure spaces for which $\mathfrak{M} \neq \phi$.

The symbol $\mathscr{P}(X)$ denotes the power set of X, $|A|$ indicates the cardinal number of the set A, the triple bar $\equiv$ is reserved for definitions and $\square$ signals the end of a proof.

DEFINITION 1.1. [2, p. 237] Let X be a set. A function $c: \mathscr{P}(X) \rightarrow \mathscr{P}(X)$ is called a *Čech closure operator* on X iff it satisfies the following three axioms:

C1: $\bar{\phi} = \phi$;