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THE LATTICE-ORDERED GROUP OF AUTOMORPHISMS OF AN α-SET

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The group of all automorphisms of a chain Ω forms a lattice-ordered group $A(\Omega)$ under the pointwise order. It is well known that if G is the symmetric group on $\aleph$ elements ($\aleph \neq 6$), then every automorphism of G is inner. Here it is shown that if Ω is an α-set, every l-automorphism of $A(\Omega)$ (preserving also the lattice structure) is inner. This is accomplished by means of an investigation of the orbits $\bar{\omega}A(\Omega)$ of Dedekind cuts $\bar{\omega}$ of Ω.

The same conjecture for arbitrary chains Ω has been investigated in [6], [4], and [8]. Lloyd proved in [6] that when Ω is the chain of rational numbers (i.e., the 0-set), or is Dedekind complete, every l-automorphism of $A(\Omega)$ is inner. He also stated this conclusion for α-sets in general, but a lacuna in his proof has been pointed out by C. Holland.

2. o-2-transitive groups $A(\Omega)$. An automorphism of a chain Ω is simply a permutation g of Ω which preserves order in the sense that $\omega < \tau$ if and only if $\omega g < \tau g$. The group $A(\Omega)$ of all automorphisms of Ω forms a lattice-ordered group (l-group) when ordered pointwise, i.e., $f \leqq g$ if and only if $\omega f \leqq \omega g$ for all $\omega \in \Omega$. We identify each $g \in A(\Omega)$ with its unique extension to $\bar{\Omega}$, the conditional completion by Dedekind cuts of Ω, and thus consider $A(\Omega)$ as an l-subgroup of $A(\bar{\Omega})$, i.e., as a subgroup which is also a sublattice.

An l-subgroup G of $A(\Omega)$ is *o-2-transitive* if for all $\beta, \gamma, \sigma, \tau \in \Omega$ with $\beta < \gamma$ and $\sigma < \tau$, there exists $g \in G$ such that $\beta g = \sigma$ and $\gamma g = \tau$. Ω is *o-2-homogeneous* if $A(\Omega)$ is o-2-transitive. (To avoid pathology, we assume throughout that Ω contains more than two points.) Corollary 16 of [8] states, for the special case in which Ω is o-2-homogeneous, that every l-automorphism ψ of $A(\Omega)$ is induced by an inner automorphism π of the larger group $A(\bar{\Omega})$, say $\pi: h \rightarrow f^{-1}hf$, f a fixed element of $A(\bar{\Omega})$; and that Ωf is an orbit $\bar{\omega}A$ of $A(\Omega)$, for some $\bar{\omega} \in \bar{\Omega}$. Thus, as was essentially obtained by Lloyd in [6] by methods different from those in [8], we have

THEOREM 1 (Lloyd). *If Ω is o-2-homogeneous, then every l-automorphism of $A(\Omega)$ is inner, provided that no orbit $\bar{\omega}A(\Omega)$, $\bar{\omega} \in \bar{\Omega}\backslash\Omega$, is o-isomorphic to Ω.*

It may be that the proviso that no orbit $\bar{\omega}A(\Omega)$, $\bar{\omega} \in \bar{\Omega}\backslash\Omega$, be o-isomorphic to Ω is satisfied by every o-2-homogeneous Ω; this is an