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THE HANF NUMBER OF OMITTING COMPLETE TYPES

SAHARON SHELAH

It is proved in this paper that the Hanf number m^C of omitting complete types by models of complete countable theories is the same as that of omitting not necessarily complete type by models of a countable theory.

Introduction. Morley [3] proved that if L is a countable first-order language, T a theory in L, p is a type in L, and T has models omitting p in every cardinality $\lambda < \beth_{\omega_1}$, *then* T has models omitting p in every infinite cardinality. He also proved that the bound $\beth_{\omega_1}$ cannot be improved, in other words the Hanf number is $\beth_{\omega_1}$. He asked what is the Hanf number m^C when we restrict ourselves to complete T and p. Clearly $m^C \leqq \beth_{\omega_1}$. Independently several people noticed that $m^C \geqq \beth_\omega$ and J. Knight noticed that $m^C > \beth_\omega$.

Malitz [2] proved that the Hanf number for complete $L_{\infty,\omega}$-theories with one axiom $\psi \in L_{\omega_1,\omega}$ is $\beth_{\omega_1}$. We shall prove

THEOREM 1. $m^C = \beth_{\omega_1}$.

NOTATION. Natural numbers will be i, j, k, l, m, n, ordinals α, β, δ; cardinals λ, μ. $|A|$ is the cardinality of A, $\beth_\alpha = \sum_{\beta<\alpha} 2^{\beth_\beta} + \aleph_0$.

M will be a model with universe $|M|$, with corresponding countable first-order language $L(M)$. For a predicate $R \in L(M)$, the corresponding relation is R^M or $R(M)$, and if there is no danger of confusion just R. Every M will have the one place predicate P and individual constants c_n such that $P = P^M = \{c_n: n < \omega\}$, $n \neq m \Rightarrow c_n \neq c_m$ (we shall not distinguish between the individual constants and their interpretation). A type p in L is a set of formulas $\varphi(x_0) \in L$; p is complete for T in L if it is consistent and for no $\varphi(x_0) \in L$ both $T \cup p \cup \{\varphi(x_0)\}$ and $T \cup p \cup \{\neg \varphi(x_0)\}$ are consistent.

An element $b \in |M|$ realizes p if $\varphi(x_0) \in p$ implies $M \models \varphi[b]$ ($\models$ -satisfaction sign), and M realizes p if some $a \in |M|$ realizes it. A complete theory in L is a maximal consistent set of sentences of L. For every permutation θ of P, model M, and sublanguage L of $L(M)$ we define an Ehrenfeucht game $EG(M, L, \theta)$ between player I and II with ω moves as follows: in the nth move first player I chooses $i \in \{0, 1\}$ and $a_n^i \in |M|$ and secondly player II chooses $a_n^{1-i} \in |M|$. Player II wins if the extension θ^* of θ defined by $\theta^*(a_n^0) = a_n^1$ preserves all atomic formulas of L. That is if $R(x_1, \cdots, x_n)$ is an atomic formula in L, $\theta^*(b_i)$ is defined then $M \models R[b_1, \cdots, b_n]$ iff $M \models R[\theta^*(b_1), \cdots, \theta^*(b_n)]$.