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COUNTEREXAMPLES IN THE BIHARMONIC CLASSIFICATION OF RIEMANNIAN 2-MANIFOLDS

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Crucial counterexamples in the biharmonic classification theory of Riemannian 2-manifolds have been deduced from certain general principles. The present note is methodological in nature: the aim is to supplement the theory by showing that very simple counterexamples can be directly constructed.

Whereas earlier work has been devoted to the class H^2 of nonharmonic biharmonic functions, here the class W of all biharmonic functions is discussed. This is of interest, since the classes O_{WB} and O_{WD} of Riemannian manifolds without (nonconstant) bounded or Dirichlet finite biharmonic functions are strictly contained in the corresponding classes O_{H^2B} and O_{H^2D}, as is seen by endowing the unit disk with a suitable conformal metric. Moreover, for W-functions the biharmonic equation need not be reduced to the Poisson equation but can be dealt with directly.

These aspects, however, are not essential. Our sole aim is to produce simple counterexamples. In particular, the function $\log\log(e^x+a)$ on a horizontal strip (Theorem 4) shows immediately that there are parabolic 2-manifolds which carry H^2D-functions. We also include some examples of 3-manifolds.

1. It is well known that there are no bounded harmonic functions on a parabolic manifold. In contrast, we shall show:

THEOREM 1. *There exist parabolic manifolds which carry nonconstant WB-functions.*

Proof. Consider in the complex (x, y)-plane the strip $\{-\infty < x < \infty; 0 \leqq y \leqq 2\pi\}$ with the lines $y = 0$ and $y = 2\pi$ identified by vertical translation so as to obtain a doubly connected Riemann surface S. The choice of the strip instead of the punctured plane is not essential, but it will slightly simplify the computation. Clearly $S \in O_G$, e.g., by virtue of the modular test (cf. [7]). On the Riemannian manifold $S_\lambda = (S, \lambda(z)\,|dz|)$ with $\lambda = e^x$, the function $u = \cos 2y$ is bounded biharmonic. In fact, $\Delta_\lambda u = e^{-2x}\Delta \cos 2y = -4\cos 2y \in H(S)$, where Δ_λ and Δ are the Laplace-Beltrami operators with respect to the metric $\lambda(z)|dz|$ and the Euclidean metric, and H stands for the class of harmonic functions. Thus $S_\lambda \in O_G - O_{WB}$.