PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 2, 1974

CONGRUENCE RELATIONS AND MULTIPLICITY TYPES OF ALGEBRAS

B. JÓNSSON AND T. P. WHALEY

Given two multiplicity types μ and μ', the following two conditions are shown to be equivalent: (1) For every algebra A of multiplicity type μ there exists an algebra A' of multiplicity type μ' such that A and A' have exactly the same congruence relations. (2) For every $k > 0$, $\mu_k + \mu_{k+1} + \cdots \leqq \mu'_k + \mu'_{k+1} + \cdots$.

Introduction. Given an algebra $A = \langle U, f_i(i \in I)\rangle$, we let Con (A) be the lattice of congruence relations over A. By the multiplicity type of A we mean the sequence $\mu = \langle \mu_0, \mu_1, \cdots, \mu_n, \cdots\rangle$ where μ_n is the number of indices i for which the rank of f_i is n. The purpose of this paper is to prove the following result:

THEOREM. *If μ and μ' are multiplicity types with the property that for every algebra A of the multiplicity type μ there exists an algebra A' of the multiplicity type μ' such that* Con (A) = Con (A'), *then $\mu_k + \mu_{k+1} + \cdots \leqq \mu'_k + \mu'_{k+1} + \cdots$ for $k = 1, 2, \cdots$.*

This result was conjectured in [3], and the special case $k = 1$ was proved in [5]. The theorem was announced in [4] for the case when the multiplicity types are finite, and in [2] without that restriction.

The converse of our theorem is also true; its proof easily reduces to an elementary but somewhat tedious set-theoretic argument. Roughly, we want to show that if the given inequalities hold, then any algebra of the multiplicity type μ can be transformed into an algebra of the multiplicity type μ' by adding dummy arguments to some of the operations and by introducing new operations that do not affect the congruence relations. (Since operations of rank zero do not affect the congruence relations, we may assume that $\mu_0 = \mu'_0 = 0$.) It is clearly sufficient to show that, given two sets I and I', partitioned into subsets I_n and I'_n, respectively, with $|I_n| = \mu_n$ and $|I'_n| = \mu'_n$ $(n = 1, 2, \cdots)$, there exists a one-to-one map $\phi: I \rightarrow I'$ such that, for each $i \in I_n$, $\phi(i)$ belongs to some I'_m with $m \geqq n$. The existence of such a map will certainly be assured if we show that I' can be partitioned into subsets J_n with $\mu_n \leqq |J_n|$ for $n = 1, 2, \cdots$ in such a way that each member of J_n belongs to some I'_m with $m \geqq n$. We shall in fact construct a double sequence of sets $J_{p,n} (p \geqq 0, n \geqq 1)$ with $J_{0,n} = I'_n$ in such a way that

(a) for a fixed p, the sets $J_{p,n}$ form a partitioning of I',