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ON A PROBLEM OF HURWITZ

NORMAN P. HERZBERG

A. Hurwitz proposed the problem of finding all the positive integers z, $\boldsymbol{x} = (x_1, \cdots, x_n)$ satisfying the diophantine equation $x_1^2 + \cdots + x_n^2 = z \cdot x_1, \cdots, x_n$. This paper investigates the question of which values of z can occur, using only the most elementary techniques. An algorithm is given for determining all permissible values of (z, n) for all n below a given bound. As an application it is established that the only possible values in the range $z \geqq (n+15)/4$ are $z = n$, $z = (n+8)/3$ when n is odd, and $z = (n+15)/4$. As another application the fifteen values of $n \leqq 131{,}020$ for which the only permissible value of z is n have been found.

2. The problem of finding all the integer solutions z, $\boldsymbol{x} = (x_1, \cdots, x_n)$ of the equation

$$x_1^2 + \cdots + x_n^2 = z \cdot x_1, \cdots, x_n \tag{1}$$

was raised by A. Hurwitz in [1]. In that paper he showed that for $n > z$ there are no solutions. This is an easy consequence of Theorem 1 (see §3) and will be replaced by the stronger result in Theorem 3. To keep this paper self-contained, let us recall the following facts from [1].

For $n = 2$, the only solutions are $z = 2$, $x_1 = x_2$; for upon setting $x_1 = dy_1$, $x_2 = dy_2$ with $(y_1, y_2) = 1$, $y_1^2 + y_2^2 = zy_1y_2$, and so $z = 2$, $x_1 = x_2 = d$.

If $z, x_1, \cdots, x_j, \cdots, x_n$ is a solution, then so is $z, x_1, \cdots, x_j', \cdots, x_n$ where x_j' satisfies

$$x_j + x_j' = z \prod_{i \neq j} x_i .$$

The n solutions derived in this way are called the *neighbors* of z, $\boldsymbol{x}$. Define the *height* of a solution to be simply $x_1 + \cdots + x_n$, and call a solution *fundamental* if its height is no greater than the height of any of its neighbors. If a solution is not fundamental, it has a neighbor of strictly smaller height, and since the heights are all positive integers, in a finite number of steps we arrive at a fundamental solution. So we see that *it suffices to study fundamental solutions.* Moreover, it obviously suffices to study solutions that satisfy

$$x_1 \geqq x_2 \geqq \cdots \geqq x_n \geqq 1 . \tag{2}$$

Also, as Hurwitz point out, it is easy to see that fundamental solutions