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THE REALIZATION OF POLYNOMIAL ALGEBRAS AS COHOMOLOGY RINGS

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To the memory of Norman Steenrod

We construct, for certain choices of a group G, a prime p, and a positive integer n, a space $X(G, p, n)$ whose cohomology ring mod p is a polynomial algebra, and we classify the polynomial algebras which can be realized as cohomology rings by this construction.

Let $\boldsymbol{Z}_p$ denote the ring of p-adic integers. From Sullivan's work on completions [15] it follows that the Eilenberg-MacLane space $K(\boldsymbol{Z}_p^n, 2)$ is the p-profinite completion of $K(\boldsymbol{Z}^n, 2)$, and that as a consequence of the p-analogue of [15, 3.9] we have

$$H^*(K(\boldsymbol{Z}_p^n, 2); Z_p) = \boldsymbol{Z}_p[x_1, x_2, \cdots, x_n]$$

where $\deg x_i = 2$. Now if G is a subgroup of $\mathrm{GL}(n, \boldsymbol{Z}_p)$ and finite, we have an action of G on the space $K(\boldsymbol{Z}_p^n, 2)$ which passes to its cohomology ring, and we define

$$X(G, p, n) = K(\boldsymbol{Z}_p^n, 2) \times_G EG$$

where EG is the total space of a universal bundle for G.

PROPOSITION. *If p does not divide the order of G, then $H^*(X(G, p, n); \boldsymbol{Z}_p)$ is the subalgebra of invariants of $H^*(K)(\boldsymbol{Z}_p^n, 2); \boldsymbol{Z}_p)$ under the action of G.*

Obviously the conclusions of this proposition apply as well with coefficients in the prime field $\boldsymbol{F}_p$ or in the field $\boldsymbol{Q}_p$ of p-adic numbers. For the sake of completeness we sketch a proof.

Proof. From [5, Th. 3.1] and [8] it follows that the cohomology of $X(G, p, n)$ is given by $\mathrm{Ext}_{Z_p(G)}(C_*(EG)), C^*(K(\boldsymbol{Z}_p^n, 2))$, where we let $\boldsymbol{Z}_p(G)$ denote the group ring over $\boldsymbol{Z}_p$ and C_* and C^* denote singular chains with coefficients in $\boldsymbol{Z}_p$. The Eilenberg-Moore spectral sequence associated with this Ext has E_2 term determined by

$$E_2^{r,s} = \mathrm{Ext}^r_{Z_p(G)}(\boldsymbol{Z}_p, H^s(K(\boldsymbol{Z}_p^n, 2); \boldsymbol{Z}_p))$$

and it follows that for $r > 0$, $|G| E_2^{r,s} = 0$ by the results of [3, Ch. XII, 2.5]. However, $E_2^{r,s}$ is a $\boldsymbol{Z}_p$-module and therefore can have only p-torsion. The fact that p does not divide $|G|$ implies that $E_2^{r,s} = 0$