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LIMIT SETS OF POWER SERIES OUTSIDE THE CIRCLES OF CONVERGENCE

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Let U denote the open unit disc in the complex plane C. A power series of a complex variable with center at the origin and radius of convergence equal to one will be called a *power series in U*. It is well-known that the behaviors of a power series outside its circle of convergence are quite irregular. In particular, it is proved that for each complex number z^*, $|z^*| > 1$, and for every closed set E in C, there is a power series in U whose limit set at z^* is E. The power series $P_\alpha(z) = \sum (1 - e^{in\alpha})z^n$, α real, are studied in this paper. Their peculiar properties seem to suggest that it might be fruitful to study the irregularities of power series outside their circles of convergence by probability theory. To study $P_\alpha(z)$, results in diophantine approximation are obtained.

We first recall the following result obtained in [2]:

THEOREM A. *There exists a power series P in U with the property that for each compact set K which lies outside the unit circle and has connected complement and for any function f continuous on K and holomorphic at the interior points of K, there exists a subsequence of the sequence of partial sums of P that converges uniformly to f on K.*

Hence, different subsequences of the sequence of partial sums of a power series in U may overconverge to different values at the same point outside the unit circle. This leads to the following

DEFINITIONS. Let P be a power series in U and let $|a| > 1$.

(1) We denote by $L(a, P)$ the set of all complex values which are the limits of all the convergent subsequences of the sequence of partial sums of P at the point a and we call it the *limit set of P at a*.

(2) If $L(a, P) \neq \emptyset$ where $|a| > 1$, we say that a is an *exceptional point* of the power series P.

For instance, if $P(z) = \sum_{k=0}^{\infty} z^k$ is the geometric series, it is clear that $L(a, P) = \emptyset$ for every point a with $|a| > 1$, so that the geometric series has no exceptional points outside the unit circle. On the other hand, if P is a "universal series" as described in Theorem A, then every point a outside the unit circle is an exceptional point of P and $L(a, P) = C$, the entire complex plane. We first establish the following result.