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POINTWISE BOUNDED APPROXIMATION BY FUNCTIONS SATISFYING A SIDE CONDITION

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In this paper necessary and sufficient conditions on a subset S of the unit disc D are given such that every bounded analytic function f on D is a pointwise limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of uniformly continuous analytic functions on D bounded by the sup norm of f and in addition satisfying $\sup\{|f_n(z)|, z \in S\} \leqq \sup\{|f(z)|, z \in S\}$ for all n.

Let $D = \{z: |z| < 1\}$ denote the open unit disc and $T = \{z: |z| = 1\}$ the unit circle. $H^\infty(D)$ denotes all bounded analytic functions on D and $A(D)$ consists of all uniformly continuous f in $H^\infty(D)$. For any $f \in H^\infty(D)$ and any subset S of D we put $\|f\|_S = \sup\{|f(z)|, z \in S\}$ and we set $\|f\| = \|f\|_D$.

A sequence $\{z_n\}_{n=1}^{\infty}$ in D converging to $z \in T$ converges *nontangentially* to z if for some constant λ we have $|z - z_n| \leqq \lambda(1 - |z_n|)$ for all n. If $f \in H^\infty(D)$ then Fatou's theorem [2, page 34] tells us that f has a nontangential limit at almost every boundary point. Thus at almost every boundary point $\lim f(z_n)$ exists and is independent of the choice of sequence. If $f \in H^\infty(D)$ is known a.e. on T we recapture its values in D by the Cauchy or Poisson integral formula. We will therefore consider functions in $H^\infty(D)$ as defined in D and a.e. on T.

A relatively closed subset S of D is called a *Farrell set* if for each $f \in H^\infty(D)$ there are $f_n \in A$, $n = 1, 2, \cdots$, converging pointwise to f on D with $\|f_n\| \leqq \|f\|$ and such that $\|f_n\|_S \leqq \|f\|_S$. This concept was introduced to us by Professor L. A. Rubel who also raised the question of describing such sets. The object here is to characterize Farrell sets in terms of their cluster points on T. The author is very grateful to Dr. A. M. Davie for valuable conversations on this subject.

First we observe that if $rz \in S$ whenever $0 < r < 1$ and $z \in S$, then S is a Farrell set. Indeed, letting $f_r(z) = f(rz)(0 < r < 1)$, we have: $f_r(z) \to f(z)$ as $r \to 1$.

On the other hand, let $S = \{z_n\}_{n=1}^{\infty}$ where $\sum_n (1 - |z_n|) < \infty$ and assume that the set of cluster points of S on T has positive linear measure. Then there are $f \in H^\infty(D)$ with $f = 0$ on S, but $f \neq 0$, while if $f \in A(D)$ and $f = 0$ on S we must have $f \equiv 0$. The set of cluster points of S on T which are not nontangential limits of sequences from S is here too large. In fact we will prove the following.

THEOREM. *A relatively closed subset S of the open unit disc D is*