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OSCILLATORY SOLUTIONS AND MULTI-POINT BOUNDARY VALUE FUNCTIONS FOR CERTAIN nth-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Consider the nth order linear differential equation

$$y^{(n)} + \sum_{k=0}^{n-1} p_k(t) y^{(k)} = 0 \; , \tag{1}$$

where $p_k(t) \in C[\alpha, \infty)$. This study explores some of the relationships between multi-point boundary value functions for (1) and the character of oscillatory solutions of (1). In particular, it is supposed for (1) that a certain $(n-1)$ point boundary value problem has no nontrivial solution and that two nontrivial solutions with $(n-1)$ zeros in common are constant multiples of each other. Under these conditions it is shown that there exists an integer i, $1 \leqq i \leqq n-1$, such that for each $a > \alpha$ and every integer l, $1 \leqq l \leqq i-1$, there is an oscillatory solution of (1) with a zero of exact multiplicity l at $t = a$. Furthermore, any solution of (1) with a zero at $t = a$ of multiplicity $l \geqq i$ is nonoscillatory.

In general, simple examples illustrate that oscillatory behavior varies widely under no additional conditions on the equation (1). In order to give some structure on which to base an investigation of the given equation, we employ multi-point boundary value functions. These functions have been studied by Alieu (in papers unavailable to the author) and by A. C. Peterson ([6], [7]). These functions were essentially used by Hanan [3] for $n = 3$ and, for $n = 4$, by Leighton and Nehari [5] as well as the author [4]. The results in this paper generalize some of the ideas of these papers.

We shall need the following definitions.

DEFINITION. We say a nontrivial solution $y(t)$ of (1) has an $i_1 - i_2 - \cdots - i_k$, $\sum_{j=1}^{k} i_j = n$, distribution of zeros on an interval I provided there exists points $t_1 < t_2 < \cdots < t_k$ in I such that $y(t)$ has a zero of order at least i_j at the point $t = t_j$, $j = 1, 2, \cdots, k$. For each $t \in [\alpha, \infty)$, $r_{i_1 i_2 \cdots i_k}(t)$ denotes the infimum of the set of numbers $b > t$ for which there exists a nontrivial solution $y(t)$ of (1) with an $i_1 - i_2 - \cdots - i_k$ distribution of zeros on $[t, b]$. If no such distribution exists, we write

$$r_{i_1 i_2 \cdots i_k}(t) = \infty \; .$$

If $r_{i_1 i_2 \cdots i_k}(t) = \infty$ for all $t \in [\alpha, \infty)$, we write