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SETS WHICH ARE TAME IN ARCS IN E^3

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Results of McMillan and Cannon may be combined to give an algebraic condition which is sufficient to show that an arc topologically embedded in E^3 is tame in E^3. The main theorem of this paper gives an essentially algebraic condition involving an arc embedded in E^3 and a compact subset of that arc which is sufficient to show that the arc may be approximated arbitrarily closely without moving the subset, to obtain a tame arc.

1. Preliminaries. The usual Euclidean distance function will be denoted by d. An open neighborhood having radius r about a set S will by denoted by $N(S, r)$. An r-set will be a set having diameter less than r.

1.1. DEFINITION. Suppose that X is a compact subset of a finite complex K which is topologically embedded in E^3. Then X is said to be tame in K iff given $r > 0$ there is a homeomorphism $h: K \rightarrow E^3$ such that

(1) $d(x, h(x)) < r$ for each x in K,
(2) $h(x) = x$ for each x in X, and
(3) $h(K)$ is tame.

1.2. DEFINITION. Suppose that X is a compact subset of an arc A which is topologically embedded in E^3. Then X is said to be untangled iff for each $r > 0$, there is an $s > 0$ such that if J is a loop in $E^3 - X$ which bounds (homologically) on an s-set in $E^3 - X$, then J shrinks (homotopically) on an r-set in $E^3 - X$.

McMillan [3] has noted that an arc is untangled iff it has free local fundamental groups (1-FLG) at each of its points. He also proved that an arc which has 1-FLG at each point is tame if each of its subarcs pierces a disk. Cannon [2, Theorem 3.16] has shown that an arc which has 1-FLG at each point does pierce a disk. Hence, an arc which is untangled is tame.

1.3. NOTATION. For the remainder of this paper A will denote an arc topologically embedded in E^3 and X will denote a compact subset of A which is untangled. The arc A will be assumed to have a fixed order, compatible with, and inducing, the given topology on A.

1.4. DEFINITION. Let Y be a subset of A. An indexed collection $C_1, \cdots, C_n$ of disjoint connected subsets of E^3 is said to be ordered