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SETS GENERATED BY RECTANGLES

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For any family F of sets, let $\mathscr{B}(F)$ denote the smallest σ-algebra containing F. Throughout this paper X denotes a set and $\mathscr{R}$ the family of sets of the form $A \times B$, for $A \subseteqq X$ and $B \subseteqq X$. It is of interest to find conditions under which the following holds:

(1) Each subset of $X \times X$ is a member of $\mathscr{B}(\mathscr{R})$

The interesting case is when

$$\omega_1 < \text{Card } X \leqq c\ ,$$

since results for other cases are known.

It is shown in Theorem 9 that (1) is equivalent to

(2) There is a countable ordinal α such that each subset of $X \times X$ can be generated from $\mathscr{R}$ is α Baire process steps .

It is also shown that the two-dimensional statements (1) and (2) are equivalent to the one-dimensional statement

(3) There is a countable ordinal α such that for each family H of subsets of X with Card $H =$ Card X, there is a countable family G such that each member of H can be generated from G in α steps .

It is shown in Theorem 5 that the continuum hypothesis (CH) is equivalent to certain statements about rectangles of the form (1) and (2) with $\alpha = 2$.

Rao [7, 8] and Kunen [2] have shown that

THEOREM 1. *If* Card $X \leqq \omega_1$ (*the first uncountable cardinal*) *then* (1) *is true and if* Card $X > c$ *then* (1) *is false.*

The question of whether (1) is true (without the requirement Card $X \leqq \omega_1$) was raised by Johnson [1] and earlier by Erdös, Ulam, and others (see [8], p. 197). The arguments in Kunen's thesis actually showed that if Card $X \leqq \omega_1$ then

(4) Each subset of $X \times X$ can be generated from $\mathscr{R}$ in 2 steps (i.e., each subset is a member of $\mathscr{R}_{\sigma\delta}$. See definitions in § 2.) .

In Theorem 5 we generalize Theorem 1 and Kunen's result (4),