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NONSOLVABLE FINITE GROUPS ALL OF WHOSE LOCAL SUBGROUPS ARE SOLVABLE, VI[1]

JOHN G. THOMPSON

This is the last paper in this series[2] and it contains the analysis of the remaining case, that is, $2 \in \pi_4$ and $e = 1$. As it happens, earlier work on this case was faulty, as I missed the group ${}^2F_4(2)$ and its simple subgroup of index 2. This lacuna is filled here, and the only change it necessitates in the earlier work is that the Main Theorem needs to be altered by added ${}^2F_4(2)'$ to the list of simple N-groups.[3]

16. The case $\mathfrak{T} \in \mathscr{M}^*$. All results in this paper are proved on the hypothesis that $2 \in \pi_4$ and $e = 1$. In this section, we also assume that if $\mathfrak{T}$ is a S_2-subgroup of $\mathfrak{G}$, then $\mathfrak{T} \in \mathscr{M}^*$. And we assume that $\mathfrak{G}$ is a minimal counterexample to the Main Theorem.

Set $\mathfrak{M} = \boldsymbol{M}(\mathfrak{T})$.

LEMMA 16.1. *If* $\mathfrak{F} \lhd \mathfrak{M}$ *and* $\mathfrak{F}$ *is an elementary abelian 2-group, then* $\mathfrak{F} \notin \mathscr{M}^*$.

Proof. Suppose false, so that $\mathfrak{M} = \boldsymbol{M}(\mathfrak{U})$ for every solvable subgroup of $\mathfrak{U}$ of $\mathfrak{G}$ which contains $\mathfrak{F}$. In particular, $\boldsymbol{C}(F) \subseteq \mathfrak{M}$ for all $F \in \mathfrak{F}^{\#}$, and also, of course $\boldsymbol{N}(\mathfrak{T}) \subseteq \mathfrak{M}$. By Lemma 13.2, there is a 2,3-subgroup $\mathfrak{H}$ of $\mathfrak{G}$ satisfying (a) through (e) of Lemma 13.2.

Let $\mathfrak{F}_0 = \mathfrak{F} \cap \mathfrak{H}$, $\mathfrak{F}_1 = \mathfrak{F} \cap \mathfrak{H}_1$, where $\mathfrak{H}_1 = \boldsymbol{O}_2(\mathfrak{H})$. Since $\mathfrak{F} \in \mathscr{M}^*$, we have $\mathfrak{F}_0 \subset \mathfrak{F}$. Since $e = 1$, $\mathfrak{H}_3$ is cyclic. Since $\boldsymbol{N}(\mathfrak{H}_2) \subseteq \mathfrak{M}$, it follows that $|\mathfrak{H}_2 : \mathfrak{H}_1| = 2$, whence $|\mathfrak{F}_0 : \mathfrak{F}_1| \leqq 2$.

If $\mathfrak{F}_0 = \mathfrak{F}_1$, then since $\mathfrak{F}_0 \subset \mathfrak{F}$, $\mathfrak{H}_1$ centralizes a subgroup $\mathfrak{E}/\mathfrak{F}_0$ of $\mathfrak{F}/\mathfrak{F}_0$ of order 2. Hence, $[\mathfrak{H}_1, \mathfrak{E}] \subseteq \mathfrak{F}_0 \subseteq \mathfrak{H}_1$, and so $\langle \mathfrak{H}_2, \mathfrak{E} \rangle$ is a 2-subgroup of $\boldsymbol{N}(\mathfrak{H}_1)$. Since Lemma 13.2(e) holds, we have $\mathfrak{E} \subseteq \mathfrak{H}_2 \cap \mathfrak{F} = \mathfrak{F}_0 = \mathfrak{F}_1$, against $|\mathfrak{E}/\mathfrak{F}_0| = 2$. Hence, $|\mathfrak{F}_0 : \mathfrak{F}_1| = 2$.

Choose $F \in \mathfrak{F}_0 - \mathfrak{F}_1$. Since $F \notin \boldsymbol{O}_2(H)$, we may assume that F normalizes $\mathfrak{H}_3$. Set $\mathfrak{K} = [\mathfrak{H}_1, \mathfrak{H}_3]$. Thus, $\mathfrak{H}_3$ has no fixed points on $\mathfrak{K}/\mathfrak{K}'$, and so

$$\mathfrak{K} = \langle [\mathfrak{K}, F], [\mathfrak{K}, F]^H \rangle ,$$

[1] An historical note is in order. In January, 1963, I announced at the meeting of the American Mathematical Society that with finitely many exceptions, the simple N-groups were $L_2(q)$ and $Sz(q)$. Had I been content to leave the explicit determination of the exceptions to someone else, I would have avoided the embarrassment of having missed ${}^2F_4(2)'$. Furthermore, several of the proofs would have been shortened considerably. But part of the fun and a great deal of the work involve pinning down the exceptions.

[2] The other papers are: Nonsolvable finite groups all of whose local subgroups are solvable, I-V: Bull. Amer. Math. Soc., 1968, Vol. 74, no. 3, pp. 383-437, Pacific J. Math., Vol. 33, no. 2, 1970, pp. 451-536, Vol. 39, no. 2, 1971, pp. 483-534, Vol. 48, No. 2, 1973, pp. 511-592, Vol. **50**, no. 1, (1974), 215-297.

[3] I have not taken the trouble to check Corollary 5 for the case $\mathfrak{G} = {}^2F_4(2)'$.