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A CHARACTERIZATION OF QF-3 RINGS

E. A. RUTTER

Let R be a ring with minimum condition on left or right ideals. It is shown that R is a QF-3 ring if and only if each finitely generated submodule of the injective hull of R, regarded as a left R-module, is torsionless. The same approach yields a simplified proof that R is quasi-Frobenius if and only if every finitely generated left R-module is torsionless.

A ring with identity is called a *left QF-3 ring* if it has a (unique) minimal faithful left module, and a QF-3 ring means a ring which is both left and right QF-3. This class of rings originated with Thrall [9] as a generalization of quasi-Frobenius or QF algebras and has been studied extensively in recent years. Quasi-Frobenius rings have many interesting characterizations and in most instances there exists an analogous characterization of QF-3 rings at least in the case of rings with minimum condition and often for a much larger class of rings. It is well known that a ring with minimum condition on left or right ideals is a left QF-3 ring if and only if the injective hull $E({}_RR)$ of the ring R regarded as a left R-module is projective. Moreover, in this case R is a QF-3 ring (cf. [6] and [8]). For semi-primary or perfect rings; however, the situation is somewhat different. Namely, a perfect ring is a left QF-3 ring if and only if $E({}_RR)$ is torsionless. A module is called *torsionless* if it can be embedded in a direct product of copies of the ring regarded as a module over itself. In this case $E({}_RR)$ need not be projective and R need not be right QF-3 (cf. [3] and [8]). However, a perfect ring is QF-3 if and only if both $E({}_RR)$ and $E(R_R)$ are projective (see [8]). In this note, it is shown that if R is left perfect ring, $E({}_RR)$ is projective if and only if each finitely generated submodule of $E({}_RR)$ can be embedded in a free R-module. For a ring with minimum condition on left or right ideals this latter condition is equivalent to each finitely generated submodule of $E({}_RR)$ being torsionless. Thus in that case QF-3 rings may be characterized by this weaker condition. The technique of proof also yields a much simplified proof of a characterization of QF rings given by the present author in [7]. Namely, a ring with minimum condition on left or right ideals is QF if and only if each finitely generated left module is torsionless. Indeed, the characterization of QF-3 rings given here may be regarded as the analog of that result.

THEOREM 1. *Let R be a left perfect ring. $E({}_RR)$ is projective if and only if each finitely generated submodule of $E({}_RR)$ can be embedded*