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SHARPENED POLYNOMIAL APPROXIMATION

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In 1951, W. Wolibner showed that a real continuous function on a closed interval can be uniformly approximated by a polynomial which interpolates at prescribed points and which has a uniform norm agreeing with the function's. This fit can be sharpened to include matching of some relative extrema as well. The paper characterizes functions that permit Simultaneous Approximation and Interpolation which is Norm-Preserving and Extrema-Matching over the entire interval except, perhaps, for a subset of arbitrarily-small diameter.

1. Introduction. The classical Weierstrass Theorem of (uniform) polynomial approximation to real, continuous functions has been generalized in many ways [2, 3]; one type of extension concerns approximations in various normed linear spaces [3]. However, if the element to be approximated remains in the space of continuous functions on a closed and finite interval, and if the uniform approximation is by polynomial, then the generalizations involve "closeness" of approximation. As an example of this second type, the Walsh result of 1935 [2] shows existence of a uniformly-good polynomial approximation to a function that also interpolates the function at prescribed points. Deutsch and Morris [3] have pointed out that W. Wolibner [4] extended this last result in 1951; he, in essence, proved existence of a polynomial which (uniformly) approximates a continuous function, and which interpolates the function, and whose (uniform) norm is equal to that of the function on the prescribed interval. Theorem 1 shows that such a fit can easily be sharpened to include matching of relative extrema too; that is, finitely-many relative extrema of the function are relative extrema of the approximating polynomial (where a minimum corresponds to a minimum and a maximum to a maximum, of course).

If f has an infinite number of relative extrema, no polynomial can match at all of them—otherwise, the derivative of the polynomial would have an infinite number of zeros. However, some continuous functions have approximating polynomials that match all extreme points except those over a closed set of arbitrarily-small diameter; Theorem 2 shows that a function has such approximating polynomials when, and only when, the points where the function achieves its extrema constitute a (countable) convergent sequence.

Throughout the discussion, the space considered is the set $C[a, b]$ of continuous, real-valued functions endowed with the max norm,