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A COUNTEREXAMPLE TO A CONJECTURE ON AN INTEGRAL CONDITION FOR DETERMINING PEAK POINTS (COUNTEREXAMPLE CONCERNING PEAK POINTS)

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Let X be a compact plane set. Denote by $R(X)$ the uniform algebra generated by the rational functions with poles off X and by $H(X)$ the space of functions harmonic in a neighborhood of X endowed with the sup norm. A point $p \in \partial X$ is a peak point for $R(X)$ if there exists a function $f \in R(X)$ such that $f(p) = 1$ and $|f(x)| < 1$ if $x \neq p$. Moreover, p is a peak point for $H(X)$ (consider Re f) and hence, by a theorem of Keldysh, p is a regular point for the Dirichlet problem. Conditions which determine whether or not a point is a peak point for $R(X)$ are thus of interest in harmonic analysis. Melnikov has given a necessary and sufficient condition that p be a peak point for $R(X)$ in terms of analytic capacity, γ; namely p is a peak point for $R(X)$ if and only if

$$\sum_{n=0}^{\infty} 2^n \gamma(A_{np} \backslash X) = \infty \ . \quad A_{np} = \left\{z: \frac{1}{2^{n+1}} \leqq |z - p| \leqq \frac{1}{2^n}\right\} .$$

Analytic capacity is generally difficult to compute, so it is desirable to obtain more computable types of conditions. Let $X^c = C \backslash X$ and

$$I = \{t \in [0, 1]: z \in X^c \text{ and } |z| = t\} .$$

In this note the following conjecture, which can be found in Zalcman's Springer Lecture Notes and which is true for certain sets X, is shown to be false in general:

Conjecture. **If $\int_I t^{-1} dt = \infty$ then 0 is a peak point for $R(X)$.**

Our counterexample uses Melnikov's theorem and the following lemma:

LEMMA. *Given $0 < a < b$ and $\log b/a < 2\pi$ there exists a set K_{ab} such that $K_{ab} \subset \{z: a \leqq |z| \leqq b\}$, $\gamma(K_{ab}) = 0$ and $\{t: z \in K_{ab}$ and $|z| = t\} = [a, b]$.*

The author is indebted to the referee for the following proof.

Garnett in [3] showed that the "Cantor corner square" set K constructed by removing all but the four corner squares of length 1/4 from the unit square, then removing all but the sixteen corner squares of length 1/16 from