PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

TOPOLOGIES ON THE TORSION-THEORETIC SPECTRUM OF A NONCOMMUTATIVE RING

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Let R-sp be the collection of all prime torsion theories on the category of left R-modules over an associative ring R. Three topologies — the order topology, the finitary order topology, and the reverse order topology (in the case that R is left noetherian) — are defined on R-sp and each is shown to exhibit some properties of the Zariski topology on the spectrum of a commutative ring. A fourth topology — the Gillman topology — is defined on R-sp when R is left noetherian and is used to characterize the separation of the reverse order topology.

1. Background and notation. Throughout the following R will always denote an associative ring with unit element 1. Unless the contrary is specifically stated, all modules and morphisms will be taken from the category R-mod of unitary left R-modules. Homomorphisms will be written as acting on the side opposite scalar multiplication, i.e., on the right. The injective hull of a module M will be denoted by $E(M)$.

The term "torsion theory" will always be used to mean hereditary torsion theory in the sense of [2]. In this section we summarize the information about torsion theories which we will need. The reader is referred to [2, 4, 6, 10] for further elucidation and for proofs.

A torsion theory τ can be completely characterized by any of the following data, each of which uniquely determines all of the others:

(i) The class $\mathscr{T}_\tau$ of torsion modules. This class is closed under taking submodules, factor modules, direct sums, and extensions (i.e., if $0 \to M' \to M \to M'' \to 0$ is an exact sequence with $M', M'' \in \mathscr{T}_\tau$, then $M \in \mathscr{T}_\tau$).

(ii) The class $\mathscr{F}_\tau$ of torsion-free modules. This class is closed under taking submodules, injective hulls, direct products, and extensions.

(iii) The set $\mathscr{L}_\tau$ of left ideals I of R satisfying $R/I \in \mathscr{T}_\tau$. This set is an idempotent filter, i.e., if $I \in \mathscr{L}_\tau$ then so does every left ideal of R properly containing I and so does $(I: r) = \{r' \in R \mid r'r \in I\}$ for every $r \in R$. Furthermore, $\mathscr{L}_\tau$ is closed under taking finite intersections and, if $I \in \mathscr{L}_\tau$ and $(H: r) \in \mathscr{L}_\tau$ for every $r \in I$ then $H \in \mathscr{L}_\tau$.

(iv) The class $\mathscr{E}_\tau$ of absolutely pure modules. These are elements N of $\mathscr{F}_\tau$ satisfying the condition that if N is a submodule of $M \in \mathscr{F}_\tau$ then $M/N \in \mathscr{F}_\tau$. The full subcategory of R-mod defined by $\mathscr{E}_\tau$ is abelian.