PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

MINIMAL SPLITTING FIELDS FOR GROUP REPRESENTATIONS

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Let T be a complex irreducible representation of a finite group G of order n and let χ be the character afforded by T. An algebraic number field $K \supset Q(\chi)$ is a splitting field for χ if T can be written in K. The minimum of $[K: Q(\chi)]$, taken over all splitting fields K of χ, is the Schur index $m_Q(\chi)$ of χ. In view of the famous theorem of R. Brauer that $Q(e^{2\pi i}/n)$ is a splitting field for χ, it is natural to ask whether there exists a splitting field L with $Q(e^{2\pi i}/n) \supset L \supset Q(\chi)$ and $[L: Q(\chi)] = m_Q(\chi)$. In this paper examples are constructed which show that such a splitting field L does not always exist. Sufficient conditions are also obtained which guarantee the existence of a splitting field L as above.

Throughout this paper Q will denote the field of rational numbers. If K is an algebraic number field and p is a prime of K, we denote the completion of K at p by K_p. If A is a simple component of a group algebra over Q, the center of A being K, and π_1 and π_2 are primes of K extending the rational prime p, then the indices of $A \otimes_K K_{\pi_1}$ and $A \otimes_K K_{\pi_2}$ are equal [2, Theorem 1]. We write $l.i._pA$ for this common value and refer to $l.i._pA$ as the p-local index of A. If $L \supset K$ and L is an abelian extension of Q, we refer to the ramification degree of a prime π of K from K to L as the q-ramification degree where π extends the rational prime q. Clearly, this does not depend on the choice of π. We use similar notation when referring to residue class degrees.

Throughout this paper χ will denote an irreducible complex character of a finite group G of order n. There is a unique constituent $\mathscr{A}$ of the group algebra of G over $Q(\chi)$ corresponding to χ in the sense that the representation of G afforded by a minimal left ideal of $\mathscr{A}$ is equivalent to $m_Q(\chi)T$, where T affords χ. If D is the division algebra component of $\mathscr{A}$ we say that D (and $\mathscr{A}$) is associated with χ. The index of D equals $m_Q(\chi)$ and χ is realizable in K if and only if K is a splitting field for D. We refer the reader to [1] for the relevant theory of algebras assumed.

We denote a primitive mth root of unity by ε_m. Gal (L/K) denotes the Galois group of L over K, and $[L: K]$ the degree of L over K. If A and B are two central simple K-algebras we write $A \sim B$ to denote that A and B are similar in the Brauer group of K.

A special case of the following lemma is proved in [6, page 631]: