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ON FINITE LEFT LOCALIZATIONS

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Of the several types of noncommutative quotient rings, finite left localizations have structure most like that of the original ring. This paper examines finite left localizations from two points of view: As rings of quotients with respect to hereditary torsion classes, and as endomorphism rings of finitely generated projective modules. In the first case, finite left localizations are shown to be the rings of quotients with respect to perfect TTF-classes. In the second, they are shown to be the double centralizers of finite projectors.

The first characterization (Corollary 1.2) shows that finite left localizations may be realized as endomorphism rings of finitely generated projective idempotent ideals. It follows that every left localization of a right perfect ring is finite.

Finally, the finite projectors of the second characterization (Corollary 2.2) are shown to have endomorphism rings whose quotient ring structure is very closely related to that of the original ring. For example, every ring of left quotients of the endomorphism ring is Morita equivalent to a ring of quotients of the original ring.

1. Perfect TTF-classes. For an associative ring R with identity let ${}_R\mathscr{M}$ denote the category of unital left R-modules. A (hereditary) torsion class $\mathscr{T} \subseteqq {}_R\mathscr{M}$ is a nonvoid class of left R-modules closed under taking submodules, factor modules, extensions, and arbitrary direct sums. Corresponding to a torsion class $\mathscr{T}$ is a torsion-free class $\mathscr{F} = \{M \in {}_R\mathscr{M} \mid \mathrm{Hom}_R(N, M) = 0\ \forall N \in \mathscr{T}\}$ and a topologizing filter of left ideals $f = \{I \text{ left ideals of } R \mid R/I \in \mathscr{T}\}$. A torsion class determines a localization functor $L_{\mathscr{T}}$ from ${}_R\mathscr{M}$ to the quotient category ${}_R\mathscr{M}/\mathscr{T}$ (see [13, § 2]) given by $L_{\mathscr{T}}(M) = \varinjlim_{I \in f} = \mathrm{Hom}_R(I, M/t(M))$ where $t(M)$ is the $\mathscr{T}$-torsion submodule of M. This localization functor is covariant, additive, and left exact, and there is a natural transformation $\sigma_M: M \to L_{\mathscr{T}}(M)$ whose kernel is $t(M)$. The object $Q = L_{\mathscr{T}}(R)$ is a ring, called the ring of left quotients of R with respect to $\mathscr{T}$, and the map σ_R is a ring homomorphism. A torsion class $\mathscr{T}$, or its ring of quotients Q, is called *faithful* if $t(R) = 0$. For more details, see [13] or [14].

Some special types of torsion classes will be of interest to us. Stenström [13, § 13] calls a topologizing filter f *perfect* in case $Q \cdot I = Q$ for every left ideal $I \in f$ (where Q is viewed as an R-module). We will call a torsion class $\mathscr{T}$ perfect if its associated filter is per-