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# ON TWO CONGRUENCES FOR PRIMALITY

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**In this paper we consider the congruences**

$$n\sigma(n) \equiv 2(\text{mod}\, \varphi(n)) , \quad \varphi(n)t(n) + 2 \equiv 0 \,(\text{mod}\, n) .$$

**1. Introduction.** Apart from the classical Wilson's theorem (that a positive integer $p > 1$ is a prime if and only if $(p-1)! + 1 \equiv 0 \,(\text{mod}\, p)$) and its variants and corollaries, there is probably no other simple primality criterion in the literature in the form of a congruence. In this connection, we may recall Lehmer's congruence [1]:

$$n - 1 \equiv 0 \,\text{mod} \phi(n) . \tag{1.1}$$

This is satisfied by every prime. We do not yet know if it has any composite $n$ as a solution. In 1932, Lehmer [1] showed that if there exists a composite number $n$ satisfying (1.1), then $n$ must be odd and square free and have at least seven distinct prime factors. This result was improved in 1944 by Fr. Schuh [4] who showed that such a $n$ must have at least eleven prime factors. In 1970, E. Lieuwens [2] corrected an error in the proof of Schuh.

In the congruences we shall consider,

$$n\sigma(n) \equiv 2(\text{mod}\, \phi(n)) \tag{1.2}$$

and

$$\phi(n)t(n) + 2 \equiv 0(\text{mod}\, n) , \tag{1.3}$$

where $\phi(n)$ is Euler's totient, and $t(n)$ and $\sigma(n)$ are respectively the number and sum of the divisors of $n$. Each of these is satisfied whenever $n$ is a prime. It is a simple matter to solve (1.2) completely (Theorem 1). However, the problem of solving (1.3) for all composite integers $n$ seems to be a deep one, and we offer only a partial solution.

**2.** THEOREM 1. *The only composite numbers $n$ satisfying* (1.2) *are* $n = 4$, 6, *and* 22.

*Proof.* Let a solution of (1.2) be

$$n = 2^a p_1^{a_1} \cdots p_r^{a_r}$$

where $p_1, \cdots, p_r$ are the distinct odd prime divisors of $n$. If for some $i (1 \leqq i \leqq r)$, $a_i > 1$, then $p_i \mid \phi(n)$ and $p_i \mid n$, so that $p_i \mid 2$, an absurdity. Hence