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SELF ADJOINT STRICTLY CYCLIC OPERATOR ALGEBRAS

MARY R. EMBRY

A *strictly cyclic* operator algebra $\mathscr{A}$ on a Hilbert space X is a uniformly closed subalgebra of $\mathscr{L}(X)$ such that $\mathscr{A}x_0 = X$ for some x_0 in X. In this paper it is shown that if $\mathscr{A}$ is a strictly cyclic self-adjoint algebra, then (i) there exists a finite orthogonal decomposition of $X, X = \sum_{j=1}^{n} \oplus M_j$, such that each M_j reduces $\mathscr{A}$ and the restriction of $\mathscr{A}$ to M_j is strongly dense in $\mathscr{L}(M_j)$ and (ii) the commutant of $\mathscr{A}$ is finite dimensional.

1. Notation and terminology. Throughout the paper X is a complex Hilbert space and $\mathscr{L}(X)$ is the algebra of continuous linear operators on X. $\mathscr{A}$ will denote a uniformly closed subalgebra of $\mathscr{L}(X)$ which is *strictly cyclic* and x_0 will be a *strictly cyclic vector* for $\mathscr{A}$: That is, $\mathscr{A}x_0 = X$. We do not insist that the identity element I of $\mathscr{L}(X)$ be an element of $\mathscr{A}$. We say that $\mathscr{A}$ is *self-adjoint* if $A^* \in \mathscr{A}$ whenever $A \in \mathscr{A}$.

If $\mathscr{B} \subset \mathscr{L}(X)$, then the *commutant* of $\mathscr{B}$ is $\mathscr{B}' = \{E: E \in \mathscr{L}(X)$ and $EB = BE$ for all B in $\mathscr{B}\}$. A closed linear subspace M of X *reduces* $\mathscr{B}$ if the projection of X onto M is in $\mathscr{B}'$. In this case M is a *minimal reducing subspace* of $\mathscr{B}$ if $M \neq \{\theta\}$ and $\{\theta\}$ is the only reducing subspace of $\mathscr{B}$ properly contained in M.

We say that a collection $\{M_j\}_{j=1}^{n}$ of closed linear subspaces of X is an *orthogonal decomposition* of X if and only if the M_j are pairwise orthogonal and span X. A collection $\{P_j\}_{j=1}^{n}$ of projections is a *resolution of identity* if and only if the collection $\{P_j(X)\}_{j=1}^{n}$ of ranges of the P_j is an orthogonal decomposition of X.

2. Introduction. Strictly cyclic operator algebras have been studied by R. Bolstein, A. Lambert, the author of this paper and others. (See for example [1], [2], and [4].) In Lemma 1 of [1] Bolstein shows that if N is a normal operator on X and $\{N\}'$ is strictly cyclic, then $\{N\}''$ is finite dimensional. This raised questions about the nature of arbitrary self-adjoint, strictly cyclic operator algebras. In this paper we show that if $\mathscr{A}$ is such an operator algebra, then there exists a finite orthogonal decomposition $\{M_j\}$ of X such that each M_j reduces $\mathscr{A}$ and $\mathscr{A}/M_j$ is strongly dense in $\mathscr{L}(M_j)$. From this it follows that $\mathscr{A}'$ is finite dimensional; indeed we show that $\mathscr{A}' = \sum_{j,k=1}^{n} P_j\mathscr{A}'P_k$ (where P_j is the projection of X onto M_j) and that for each j and k, $P_j\mathscr{A}'P_k$ is of dimension zero or one. If $\mathscr{A}'$