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# ON THE MAXIMUM AND MINIMUM OF PARTIAL SUMS OF RANDOM VARIABLES

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**Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variable with $S_k = X_1 + X_2 + \cdots + X_k$ and $S_k^+ = \max[0, S_k]$. We shall derive formulas for the computation of $E\left[\min_{1\leq k\leq n} S_k^+\right]$, $E\left[\max_{1\leq k\leq n} S_k\right]$, and $E\left[\min_{1\leq k\leq n} S_k\right]$. The formulas are then applied to the case of standard normal random variables.**

**2. Notation, definitions, and preliminary lemmas.** Let $x = (x_1, \cdots, x_n)$ be a vector with real components and $-x = (-x_1, \cdots, -x_n)$. In some instances we shall also assume that the components $x_i$, $i = 1, 2, \cdots, n$ are rationally independent. (For rational $r_i$, $r_1x_1 + r_2x_2 + \cdots + r_nx_n = 0$ if and only if each $r_i = 0$.) Let $x_{k+n} = x_k$, and $x(k) = (x_k, x_{k+1}, \cdots, x_{k+n-1})$, $k = 1, 2, \cdots, n$. Let $s_k = x_1 + x_2 + \cdots + x_k$. Call the polygon connecting the points $(0, 0)$, $(1, s_1), \cdots, (k, s_k), \cdots, (n, s_n)$ the sum polygon of the vector $x$, and the line connecting $(0, 0)$ with $(n, s_n)$ the chord of the sum polygon. The sum polygon for the cyclically permuted vector $x(k)$ is defined the same way.

F. Spitzer proved in [2] (see especially page 325, lines 8–12) the following lemma.

LEMMA 1. *Let $x = (x_1, \cdots, x_n)$ be a vector such that the components $x_i$, $i = 1, \cdots, n$, are rationally independent. Consider the sum polygons of the $n$ cyclic permutations of $x$ and prescribe an integer $r$ between 0 and $n - 1$. The sum polygon of exactly one of the cyclic permutations of $x$ has the property that exactly $r$ of its vertices lie strictly above its chord.*

We adopt the following notations and definitions. For any real $a$,

$$a^+ = \max[0, a], \qquad a^- = \min[0, a] \tag{2.1}$$

$\sigma$ is the permutation on $n$ symbols, so that

$$\sigma x = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix} x = (x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}) . \tag{2.2}$$

We write