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THE NONMINIMALITY OF THE DIFFERENTIAL CLOSURE

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The differential closure of a given ordinary differential field k is characterized to within (differential) k-isomorphism as a differentially closed (differential) extension field $\hat{k}$ of k which is k-isomorphic to a subfield of any differentially closed extension field of k. It has been conjectured that, in analogy to the cases of the algebraic closure of a field and the real closure of an ordered field, the differential closure of any differential field k is minimal, that is, not k-isomorphic to a proper subfield of itself. The conjecture is here shown to be false.

Let k be a differential field (ordinary, that is with one specified derivation) of characteristic zero and let $k\{y\}$ be the differential ring of differential polynomials over k in the differential indeterminate y. Recall that the *order* of a nonzero differential polynomial in $k\{y\}$ is simply the smallest integer $r \geqq -1$ such that the differential polynomial involves none of the derivatives $y^{(r+1)}, y^{(r+2)}, \cdots$. According to Lenore Blum's definition, k is *differentially closed* if, for any $f, g \in k\{y\}$ with g of smaller order than f, there is a zero of f in k that is not a zero of g. For any differential field k, a *differential closure* of k is a differential extension field $\hat{k}$ of k that is differentially closed and that can be k-embedded in any differentially closed differential extension field of k. Blum has used the methods of model theory to show the existence of $\hat{k}$ and to derive a number of its properties [2], appreciably extending and simplifying a theory initiated by Abraham Robinson [5]. The uniqueness of $\hat{k}$ to within differential k-isomorphism follows from a recent result of Shelah [7]. The differential closure $\hat{k}$ of k is called *minimal* if there is no (differential) k-isomorphism of $\hat{k}$ with a proper subfield of itself. One of the unsolved problems of the theory has been to determine whether or not $\hat{k}$ is always minimal. Sacks has conjectured [6] that $\hat{k}$ is minimal over k in the special case $k = \boldsymbol{Q}$. It is proved here, among other things, that this conjecture is false. It was learned after the completion of this paper that this result has also been proved by Kolchin [4] and announced by Shelah [8]. The author is greatly indebted to Lenore Blum for calling his attention to the problem and for numerous conversations on her work.

We begin by recalling some facts outlined in a recent paper of Ax [1]. Let $k \subset K$ be fields. There is a K-module $\Omega^1_{K/k}$, the space of differential forms of degree one of K/k, and a k-linear map $d: K \rightarrow \Omega^1_{K/k}$ such that $d(xy) = xdy + ydx$ for all $x, y \in K$ (and these can be